CHAPTER ONE
PERCENTAGE ERROR (P.E) AND RATIO

AIM: The overall aim of this lecture is to solve problems of Percentage Error (P.E), Ratio/Proportion and Rates.

OBJECTIVES: At the end of this lecture students should be able to:

1. Explain the meaning of Percentage error and solve problems involving Percentage Error. (P.E).
2. Explain the meaning of Ratio/Proportion and Solve Problems involving Ratio/Proportion.
3. Explain the meaning of Rates and solves problems involving Rates.

1.1 Percentage Error

Percentage error is the result of little degree of error allowed in measurement.

$$Percentage\ error = \frac{Wrong\ value - Actual\ value}{Actual\ value} \times \frac{100}{1}$$

$$Percentage\ error = \frac{Error}{Exact} \times \frac{100}{1}$$

Example 1: Instead of recording the number 1.21cm of a tube, a nurse recorded 1.42cm. Find the percentage error correct to one decimal place.

Solution:

Wrong value = 1.42cm, Actual value = 1.21cm

$$P.E = \frac{1.42-1.21}{1.21} \times 100 = \frac{0.21}{1.21} \times 100 = 17.36\%$$

Example 2: A student measured the length of a room and obtained the measurement as 4.99m. If the percentage error of his measurement is 5% and his own measurement was smaller than the length. What is the length of the room.

Solution:

Let the actual length be x

Wrong value = 4.99m (4.99m < x)

$5\% = \frac{x-4.99}{x} \times 100$ Divide both sides by 100.

$\frac{5}{100} = \frac{x-4.99}{x}$ Cross Multiply

$5x = 100x - 499$ Collect like terms

$499 = 100x - 5x$

$499 = 95x$ Divide both sides by 95

$\frac{499}{95} = x$

$\boldsymbol{x = 5.25m}$ **This is the length of the room**

Example3: if the percentage error of a lab scientist during a laboratory test is 0.05% and the actual measurement is 20cm. Calculate the recorded measurement.

Solution:

$P.E = \frac{Error}{Exact} \times 100$

$0.05 = \frac{Error}{20} \times 100$

$0.05 \times 20 = 100 \times Error$

$Error = \frac{0.05 \times 20}{100}$

$Error = 0.01cm$

Recorded measurement = 20 + 0.01 = 20.01cm

Assessment 1:

1. A boy estimated his transport fare for a journey as N190 instead of N200. Solve the percentage error in his estimate

   A. 95% B. 47.5% C. 5.26% D. 5%

2. Solve correct to two significant figures, the percentage error in approximating 0.375 to 0.4

   A. 2.0 B. 2.5 C. 6.6 D. 6.7

3. The weight of a baby is measured to be 3.20kg. If its actual length is 3.25m, what is the percentage error

   A. 1.5% B. 1.8% C. 2.0% D. 2.3%

1.2 Ratio

This is the act of comparing like quantities such as weights, lengths, money e.t.c

Ratio $m : n$ is represented in fraction $\frac{m}{n}$

Example 4: Express the ratio 50k : N2.50 as simply as possible.

Solution:

$$\frac{50K}{N2.5} = \frac{50K}{2.5\times100} = \frac{50}{250} = \frac{1}{5} = 1K:5K$$

Example 2: The ratio of the number of men to the number of women of 20 patients receiving treatment in a hospital is 4:1. Solve

a) The number of male patients.

b) The number of female patients.

Solution:

Ratio (Men : Women = 4 : 1)

Total Ratio number = 4 + 1 = 5.

a) Number of Male patient $= \frac{4}{5} \times 20 = \frac{80}{5} = 16$

b) Number of Female patient $= \frac{1}{5} \times 20 = \frac{20}{5} = 4$

Example 3: If 3 boys shared x oranges among themselves in the ratio 3 : 5 : 8 and the smallest share was 90 oranges. Solve the value of *x*.

Solution:

Sum of the ratio = 3 + 5 + 8 = 16

Smallest value of the ratio is 3

$\frac{3}{16} \times x = 90$ Cross Multiply

$3x = 16 \times 90$ Divide both sides by 3

$x = \frac{16 \times 90}{3} = 16 \times 30 = 480$

Hence, 480 oranges was shared among themselves

Example 4: Tope, Ebi and Rukevwe agreed to share the expenses for an evening outing in the ratio 3 : 4 : 5 respectively. Tope paid N200.00 for taxi, Ebi paid

N500.00 for food and Rukevwe paid N300.00 for Cinema. How much should Tope and Rukevwe pay Ebi to settle their agreed Share.

Solution:

Ratios	Tope	:	Ebi	:	Rukevwe
	3	:	4	:	5

Total ratio = 3 + 4 + 5 = 12

Total Expenses = 200 +500 + 300 = N1,000

Agreed ratio Share:	Tope	Ebi	Rukevwe
=	$\frac{3}{12} \times 1000$	$\frac{4}{12} \times 1000$	$\frac{5}{12} \times 1000$
=	N250	N333	N417
Actual Amount =	N200	N500	N300
Difference in Amount =	N50	N167	N117

Therefore Tope **will have to pay N117 to** Ebi **summing up N167 to settle the agreed Share**

Example 5: if $x : y = 2 : 3$ and $y : z = 4 : 7$, Find $x : y : z$

Solution:

$x : y = 2 : 3$ …….(1)$\times 4$

$y : z = 4 : 7$ ……..(2)$\times 3$

$x : y = 8 : 12$

$y : z = 12 : 21$

$x : y : z = 8 : 12 : 21$

Assessment:

1. If a : b = 4 : 9 and b : c = 3 : 5, Find a : b : c

A. 2:9:15 B. 3:9:14 C. 4:9:15 D. 4:9:20 E. 4:12:45

2. Divide N738.00 in the ratios of $1\frac{1}{2} : 2 : 3\frac{1}{3}$. What is the highest amount

A. N152 B. N162 C. N216 D. N320 E. N360

3. The sum of N240 is shared between Kwesi, Usman and Samuel. If Kwesi received N60 and the remaining amount is shared between Usman and Samuel in the ratio 4:5 respectively. How much does Samuel receive?

A. N80 B. N90 C. N100 D. N120

CHAPTER TWO

REAL NUMBERS, LOGARITHM, SURDS AND INDICES

2.1 Real numbers

These are numbers that can be used to measure a one-dimensional quantity that is continuous. For example, distance, temperature, time etc. real numbers are referred to as rational or irrational number and are either positive or negative integers usually expressed mathematically as R

2.1.1 Rational numbers: These are numbers expressed as $\frac{a}{b}$, where $b \neq 0$. For example $\frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \ldots$

Integers are rational numbers.

2.1.2 Integers: These are whole numbers that are real and positive or negative. For example ..,-3,-2,-1,0,1,2,3,…

Whole numbers are integers

2.1.3 Whole numbers: These are collection of natural numbers together with a zero. For example 0,1,2,3,4…..

Natural numbers are whole numbers

2.1.4 Natural numbers: These are numbers that starts from 1, containing all counting numbers. For example, N = {1,2,3,4,}

2.1.5 Irrational number: these are numbers expressed as $\sqrt{a}$. where $a \neq 0$. For example $\sqrt{2}$, $\frac{\sqrt{2}}{3}, \frac{1}{\sqrt{2}}$ $etc.$

2.1.6 Properties of Real Numbers

Real Numbers exhibits properties such as

I. Commutative
II. Associative
III. Distributive
IV. Identity

Commutative:

$a + b = b + a$. For example, $2 + 3 = 3 + 2 = 5$

$a \times b = b \times a$. For example, $2 \times 3 = 3 \times 2 = 6$

Associative:

$a + (b + c) = (a + b) + c$. For example $1 + (2 + 3) = (1 + 2) + 3 = 6$

$(ab)c = a(bc)$. For example, $(2 \times 3) \times 4 = 2 \times (3 \times 4) = 24$

Distributive:

Right distributive: $a \times (b + c) = ab + ac$. For example $2 \times (3 + 4) = 6 + 8 = 14$

Left Distributive: $(a + b) \times c = ac + bc$. For example $(2 + 3) \times 4 = 8 + 12 = 20$

Identity:

$a + 0 = a$. Where 0 is the additive identity element

$a \times 1 = 1 \times a = a$. Where 1 is the multiplicative identity element

Evaluate $5 \times 5 - 5 \div 5 + 5$

A complex number is a combination of the real and imaginary number. This will looked at in later chapter in this text book

2.2 Logarithm

Logarithm was introduced in the 17^{th} century by John Napier. Logarithm could be defined as the order to which a number is raised to get other values

Logarithm of $a^x = b$ is expressed in the form $log_a b = x$

Logarithm can be common or Natural. Logarithm numbers are naturally in base 10 i.e. $log_{10} a$

Common: These Logarithms are in the base of 10. For example, $log_{10} 1000 = log_{10} 10^3 = 3log_{10} 10 = 3$

Natural: These Logarithms are in the base of e. For example, $log_e 50 = 1.699$

Exercise: what is the expression for $Log_e 90$, $Log_{10} 10000$

2.2.1 Laws of Logarithm

1. **Product Law:** $Log_a xy = Log_a x + Log_a y$

   For Example, $Log_3 30 = Log_3(3 \times 10) = Log_3(3) + Log_3(10) = 1 + Log_3(10)$

2. **Division Law:** $Log_a(x \div y) = Log_a x - Log_a y$

   For Example, $Log_3(3 \div 10) = Log_3(3) - Log_3(10) = 1 - Log_3(10)$

3. **Exponential Law:**

   $Log_a b^x = xLog_a b$

   Example: $Log_2 8 = Log_2 2^3 = 3Log_2 2 = 3$

$Log_a b = {}^{Log_c b}/_{Log_c a}$

$Log_a b = {}^{1}/_{Log_b a}$

4. **Derivative Law:** $y = Log_e x$, then $\frac{dy}{dx} = \frac{1}{x}$; $y = Log_{10} x$, then $\frac{dy}{dx} = \frac{1}{xIn(10)}$
5. **Integration Law:** $\int Log_a x \, dx = x\left(Log_a x - \frac{1}{In(a)}\right) + c$
6. **Identity Law:** $Log_a a = 1$
7. **Zero Law:** $Log_a 1 = 0$

Other Laws

8. $Log_a(xy) = yLog_a(x)$
9. $Log_a(y\sqrt{x}) = Log_a\left(\frac{x}{y}\right)$
10. $sLog_a x + tLog_a y = Log_a(x^s y^t)$
11. $Log_a(x + y) = Log_a x + Log_a(1 + xy)$
12. $Log_a(x - y) = Log_a x + Log_a(1 - xy)$

Example 1: solve x in the expression $Log_x 27 = 3$

Solution

$Log_x 27 = 3$. This implies that $27 = x^3$

$3^3 = x^3$

$x = 3$

Example 2: Evaluate $Log_5 125 = x$

Solution

$Log_5 125 = x$. This implies that $125 = 5^x$

$5^3 = 5^x$

$x = 3$

Example 3: Evaluate $2Log_3 x = 2 + Log_3 4$

Solution

$2Log_3 x = 2 + Log_3 4$

$Log_3 x^2 = 2Log_3 3 + Log_3 4$

$Log_3 x^2 = Log_3 3^2 + Log_3 4$

$Log_3 x^2 = Log_3 (3^2 \times 4) = Log_3 (9 \times 4)$

$Log_3 x^2 = Log_3 (9 \times 4)$

$x^2 = 36$ => $x = 6$ We obtained 6 as the answer by taking the square root of both sides

Example 4: Evaluate $Log_3 \left(\frac{1}{27}\right)$

Solution

Applying the properties of Logarithm,

$Log_3 \left(\frac{1}{27}\right) = Log_3 1 - Log_3 27$

$= 0 - Log_3 27$

$= -Log_3 3^3 = -3Log_3 3$

$= -3 \times 1 = -3$

Alternatively,

$Log_3\left(\frac{1}{27}\right) = Log_3\left(\frac{1}{3^3}\right) = Log_3 3^{-3}$

$= -3Log_3 3 = -3 \times 1 = -3$

2.3 Surds

Any number that cannot have a square root as a whole number is referred to as a surd

$\sqrt{4}, \sqrt{9}, \sqrt{16},$ .... are not called surds

$\sqrt{2}, \sqrt{3}, \sqrt{7},$ .... are called surds

Given that $y = \sqrt[n]{x}, n$ is called the order of x , y and x is the radicand of y

2.3.1 Classification of Surds

Surds are classified as

I. Pure
II. Mixed
III. Compound

Pure surds are made up of irrational numbers alone. For example $\sqrt{2}, \left(\sqrt{3}\right)^5, \sqrt[5]{7},$ etc.

Mixed surds are made up of both rational and irrational numbers. For example $7\sqrt{2}, 5\sqrt{3}, \ldots$

Compound surds contain two or more surds. For example $7\sqrt{2} + \sqrt{5}, \sqrt{3} + \sqrt{5}, \ldots.$

2.3.2 Properties of Surd

Surd Properties include

1. Addition or Subtraction: $\sqrt{x} \pm \sqrt{y} \neq \sqrt{x \pm y}$
2. Multiplication: $\sqrt{x} \times \sqrt{y} = \sqrt{x \times y}$
3. Division: $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}} \;\; => \;\; \frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$

   Rationalization:

$$\frac{a+\sqrt{x}}{b+\sqrt{y}} = \frac{(a+\sqrt{x})(b-\sqrt{y})}{(b+\sqrt{y})(b-\sqrt{y})} = \frac{ab-a\sqrt{y}+b\sqrt{x}-\sqrt{xy}}{b^2-y}$$

4. Exponential : $\sqrt{x} = (x)^{\frac{1}{2}}$ Also $\sqrt[a]{x} = (x)^{\frac{1}{a}}$; $\left(\sqrt[a]{x}\right)^b = (x)^{\frac{b}{a}}$

Example 5: solve the Surd $\sqrt{300} + \sqrt{27}$

Solution

$\sqrt{300} + \sqrt{27} = \sqrt{100 \times 3} + \sqrt{9 \times 3}$

$= 10\sqrt{3} + 3\sqrt{3} = 13\sqrt{3}$

Example 6: Multiply $\sqrt{300}\ by\ \sqrt{27}$

Solution

$\sqrt{300} \times \sqrt{27} = \sqrt{100 \times 3} \times \sqrt{9 \times 3}$

$= 10\sqrt{3} \times 3\sqrt{3} = 30\sqrt{3 \times 3} = 30 \times 3$

$= 90$

Alternatively,

$\sqrt{300} \times \sqrt{27} = \sqrt{300 \times 27} = \sqrt{8100} = 90$

Example 7: Evaluate $\left(3 + 2\sqrt{3}\right)^2$

Solution

Applying Pascal triangle expansion

$\left(3 + 2\sqrt{3}\right)^2 = 3^2 + \left(3 \times 2 \times 2\sqrt{3}\right) + \left(2\sqrt{3}\right)^2$

$= 9 + 12\sqrt{3} + (4 \times 3) = 9 + 12 + 12\sqrt{3}$

$= 21 + 12\sqrt{3}$

Example 8: Evaluate $\left(3\sqrt{2} + 2\sqrt{3}\right) + \left(6\sqrt{2} - 5\sqrt{3}\right)$

Solution

$\left(3\sqrt{2} + 2\sqrt{3}\right) + \left(6\sqrt{2} - 5\sqrt{3}\right) = (3 + 6)\sqrt{2} + (2 -$

$5)\sqrt{3}$

$= 9\sqrt{2} - 3\sqrt{3}$

Example 9: Evaluate $\left(3\sqrt{2} + 2\sqrt{3}\right) \times \left(6\sqrt{2} - 5\sqrt{3}\right)$

Solution

$\left(3\sqrt{2}+2\sqrt{3}\right)\times\left(6\sqrt{2}-5\sqrt{3}\right)=3\sqrt{2}\left(6\sqrt{2}-5\sqrt{3}\right)+2\sqrt{3}\left(6\sqrt{2}-5\sqrt{3}\right)$

$\left(18\sqrt{4}-15\sqrt{6}\right)+\left(12\sqrt{6}-10\sqrt{9}\right)=18(2)-15\sqrt{6}+12\sqrt{6}-10(3)$

$=36-30-3\sqrt{6}=6-3\sqrt{6}$

2.3.3 Rationalization of Surd

This is the process of multiplying a surd by another surd i.e. $\sqrt{2}\times\sqrt{2}=2$

$\left(6\sqrt{2}-5\sqrt{3}\right)$ when multiplied by $\left(6\sqrt{2}+5\sqrt{3}\right)$ is rationalized to $6\sqrt{2}\left(6\sqrt{2}+5\sqrt{3}\right)-5\sqrt{3}\left(6\sqrt{2}+5\sqrt{3}\right)=36\sqrt{4}+30\sqrt{6}-30\sqrt{6}-25\sqrt{9}=36(2)-25(3)=72-75=-3$

Example 9: Evaluate $\left(3\sqrt{2}+2\sqrt{3}\right)\big/\left(6\sqrt{2}-5\sqrt{3}\right)$

Solution

$\left(3\sqrt{2}+2\sqrt{3}\right)\big/\left(6\sqrt{2}-5\sqrt{3}\right)=\frac{\left(3\sqrt{2}+2\sqrt{3}\right)\left(6\sqrt{2}+5\sqrt{3}\right)}{\left(6\sqrt{2}-5\sqrt{3}\right)\left(6\sqrt{2}+5\sqrt{3}\right)}$

$=\frac{3\sqrt{2}\left(6\sqrt{2}+5\sqrt{3}\right)+2\sqrt{3}\left(6\sqrt{2}+5\sqrt{3}\right)}{6\sqrt{2}\left(6\sqrt{2}+5\sqrt{3}\right)-5\sqrt{3}\left(6\sqrt{2}+5\sqrt{3}\right)}=\frac{18\sqrt{4}+15\sqrt{6}+12\sqrt{6}+10\sqrt{9}}{36\sqrt{4}+30\sqrt{6}-30\sqrt{6}-25\sqrt{9}}$

$$= \frac{18(2)+27\sqrt{6}+10(3)}{36(2)-25(3)} = \frac{36+30+27\sqrt{6}}{72-75}$$

$$= -\frac{6+27\sqrt{6}}{3} = -2 - 9\sqrt{6}$$

2.3.4 Quadratic Surd

$$\sqrt{(m+\sqrt{n})} = \sqrt{a} + \sqrt{b}$$

Take the square root of both sides,

$$\left(\sqrt{(m+\sqrt{n})}\right)^2 = \left(\sqrt{a}+\sqrt{b}\right)^2$$

$$\left(m+\sqrt{n}\right) = a + b + 2\sqrt{ab}$$

$$\left(m+\sqrt{n}\right) = a + b + 2\sqrt{ab}$$

From the above equation, $a + b = m; ab = n$

$(a-b)^2 = a^2 + b^2 - 2ab$ ....*

$(a+b)^2 = a^2 + b^2 + 2ab$ ....**

$a^2 + b^2 = (a+b)^2 - 2ab$ ....***

Substitute *** into *

$$(a-b)^2 = (a+b)^2 - 2ab - 2ab$$

$$(a-b)^2 = (a+b)^2 - 4ab$$

Example 9: Evaluate $the\ square\ root\ of\ \left(5+\sqrt{3}\right)$

Solution

Let $\sqrt{(5+\sqrt{3})} = \sqrt{a} + \sqrt{b}$

Take the square root of both sides,

$$\left(\sqrt{(5+\sqrt{3})}\right)^2 = \left(\sqrt{a} + \sqrt{b}\right)^2$$

$$(5+\sqrt{3}) = a + b + 2\sqrt{ab}$$

$$(5+\sqrt{3}) = a + b + 2\sqrt{ab}$$

From the above equation, $a + b = 5; ab = 3$

$(a-b)^2 = a^2 + b^2 - 2ab$ ....*

$(a+b)^2 = a^2 + b^2 + 2ab$ ....**

$a^2 + b^2 = (a+b)^2 - 2ab$ ....***

Substitute *** into *

$$(a-b)^2 = (a+b)^2 - 2ab - 2ab$$

$$(a-b)^2 = (a+b)^2 - 4ab$$

$$(a-b)^2 = 5^2 - 4(3) = 25 - 12 = 13$$

$a - b = \pm\sqrt{13}$ ....(1)

$a + b = 5$ ....(2)

$$2a = 5 + \sqrt{13}$$

$a = \frac{5+\sqrt{13}}{2}$ or $a = \frac{5-\sqrt{13}}{2}$

Recall $ab = 3$

$\frac{5\pm\sqrt{13}}{2}\ b = 3$

$b = \left(\frac{6}{5+\sqrt{13}}\right)$ or $b = \left(\frac{6}{5-\sqrt{13}}\right)$

$b = \frac{30-6\sqrt{13}}{12}\ or\ b = \frac{30+6\sqrt{13}}{12}$

$$\sqrt{(5+\sqrt{3})} = \sqrt{\frac{5\pm\sqrt{13}}{2}} + \sqrt{\frac{30\pm6\sqrt{13}}{12}}$$

$$\sqrt{(5+\sqrt{3})} = \sqrt{\frac{5\pm\sqrt{13}}{2}} + \sqrt{\frac{5\pm\sqrt{13}}{2}} = 2\left(\sqrt{\frac{5\pm\sqrt{13}}{2}}\right)$$

2.4 Indices

Let b, m, n be real numbers where $b^m = n$, then n is called the order or power of b, b is the base and m is the index, called indices in plural form express as exponents or power.

Examples of indices are $2^4, 3^{\frac{1}{4}}$, ...

2.4.1 Laws of Indices

The laws of indices are given below

1. Product rule: $a^x \times a^y = a^{x+y}$

   Example: $3^2 \times 3^3 = 3^{2+3} = 3^5 = 243$

2. Division rule: $\frac{a^x}{a^y} = a^{x-y}$

   Example: $\frac{3^5}{3^2} = 3^{5-2} = 3^3 = 27$

3. Exponential rule; $(a^x)^y = a^{xy}$; $(ab)^x = a^x b^x$

   Example: $(3^2)^3 = 3^6 = 729$

   $(3 \times 2)^3 = 3^3 \times 2^3 = 27 \times 8 = 216$

4. Identity rule: $a^0 = 1$; $a^1 = a$

   Example: $3^0 = 1, 100^0 = 1$

5. Inverse Power:

   $(a)^{\frac{x}{y}} = \left(\sqrt[y]{a}\right)^x$

   $(a)^{\frac{1}{x}} = \sqrt[x]{a}$

   $(a)^{-\frac{1}{x}} = \frac{1}{(a)^{\frac{1}{x}}} = \frac{1}{\sqrt[x]{a}}$

   $(a)^{-x} = \frac{1}{(a)^x}$

Example 10: Simplify $\frac{(a^{2x+y})(a^{2x-3y})}{a^{x+2y}}$

Solution

$$\frac{(a^{2x+y})(a^{2x-3y})}{a^{x+2y}} = \frac{a^{2x+y+2x-3y}}{a^{x+2y}} = \frac{a^{4x-2y}}{a^{x+2y}}$$

$$= a^{4x-2y-(x+2y)} = a^{4x-2y-x-2y} = a^{3x-4y}$$

Example 10: Simplify $\left(\frac{27}{64}\right)^{-\frac{2}{3}}$

Solution

$$\left(\frac{27}{64}\right)^{-\frac{2}{3}} = \left(\frac{3}{4}\right)^{3\left(-\frac{2}{3}\right)} = \left(\frac{3}{4}\right)^{-2}$$

$$= \frac{1}{\left(\frac{3}{4}\right)^2} = \frac{1}{\frac{9}{16}} = \frac{16}{9}$$

Example 11: Given that $3^a = 9^b = 27^c$ *and* $\frac{1}{3a} + \frac{1}{9b} + \frac{1}{27c} = \frac{1}{9}$, what is the value of a, b and c

Solution:

$$3^a = 9^b = 27^c \text{ and } \frac{1}{3a} + \frac{1}{9b} + \frac{1}{27c} = \frac{1}{9}$$

$$3^a = 9^b => 3^a = 3^{2b} => a = 2b => b = \frac{a}{2}$$

$$3^a = 27^c => 3^a = 3^{3c} => a = 3c => c = \frac{a}{3}$$

$$\frac{1}{3a} + \frac{1}{9\left(\frac{a}{2}\right)} + \frac{1}{27\left(\frac{a}{3}\right)} = \frac{1}{9}$$

$$\frac{1}{3a} + \frac{2}{9a} + \frac{1}{9a} = \frac{1}{9}$$

$$\frac{3+2+1}{9a} = \frac{1}{9}$$

$$\frac{6}{9a} = \frac{1}{9} => a = \frac{1}{6}$$

$$b = \frac{a}{2} => b = \frac{1}{2(6)} = \frac{1}{12}$$

$c = \frac{a}{3} = \frac{1}{3(6)} = \frac{1}{18}$

Assignment

1. Evaluate $log_5 15625$
2. solve for x in the expression $Log_x 64 = 2$
3. Evaluate $Log_3 9 = x$
4. Evaluate $2Log_5 x = 2 + Log_5 7$
5. Evaluate $Log_2 \left(\frac{1}{8}\right)$
6. solve the Surd $\sqrt{500} + \sqrt{80}$
7. Evaluate $\left(4 + 2\sqrt{5}\right)^2$
8. Evaluate $\left(5\sqrt{2} + 2\sqrt{5}\right) + \left(\sqrt{2} - 5\sqrt{5}\right)$
9. Evaluate $\left(5\sqrt{2} + 2\sqrt{5}\right) - \left(\sqrt{2} - 5\sqrt{5}\right)$
10. Evaluate $\left(5\sqrt{2} + 2\sqrt{5}\right)\left(\sqrt{2} - 5\sqrt{5}\right)$
11. Evaluate $\frac{\left(5\sqrt{2}+2\sqrt{5}\right)}{\left(\sqrt{2}-5\sqrt{5}\right)}$
12. Evaluate the square root of $\left(3 + \sqrt{5}\right)$
13. Evaluate the square root of $\left(\sqrt{32} - \sqrt{24}\right)$
14. Given that $2^a = 4^b = 16^c$ *and* $\frac{1}{2a} + \frac{1}{4b} + \frac{1}{16c} = 100$, what is the value of a, b and c

15. Simplify $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$

16. Simplify $\frac{(a^{5x+2y})(a^{x-7y})}{a^{3x+2y}}$

CHAPTER THREE
QUADRATIC EQUATION.

AIM: The overall aim of this lecture is to solve problems of quadratic functions.

OBJECTIVES: At the end of this lecture students should be able to:

1. Use algebraic methods to solve problems involving quadratic equation.
2. Use sum and products of a root to solve a quadratic function.

3.1 Introduction

$ax^2 + bx + c = 0$ Where a, b and c are constants such that $a \neq 0$

This is the general form for a quadratic equation.

3.2 Transforming the Quadratic Equation to Quadratic Formula using Completing the Square Method.

$$ax^2 + bx + c = 0$$

$$ax^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + (\frac{1}{2} \times \frac{b}{a})^2 = -\frac{c}{a} + (\frac{1}{2} \times \frac{b}{a})^2$$

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac+b^2}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac+b^2}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{-4ac+b^2}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

This is the general formula for a quadratic equation called the Almighty formula.

3.3 Discriminant

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

Where $\boldsymbol{D = b^2 - 4ac}$. This is called the **Discriminant.**

When $D > 0$ the roots are **real and different**

When $D = 0$ the roots are **real and equal**

When $D < 0$ the roots are **real and complex**

3.4 Sum and Product of Roots

3.4.1 Discriminant Method

Show that the sum of the roots $(\alpha + \beta) = -\frac{b}{a}$ and the product of the roots $\alpha\beta = \frac{c}{a}$ using the discriminant method.

$\alpha = \frac{-\text{b}+\sqrt{D}}{2a}$ and $\beta = \frac{-\text{b}-\sqrt{D}}{2a}$

$\alpha + \beta = \frac{-\text{b}+\sqrt{D}}{2a} + \frac{-\text{b}-\sqrt{D}}{2a}$

$= \frac{-2b}{2a}$

$= -\frac{b}{a}$

$\alpha\beta = \frac{-\text{b}+\sqrt{D}}{2a} \times \left(\frac{-\text{b}-\sqrt{D}}{2a}\right)$

$= \frac{b^2-D}{4a^2}$

$= \frac{b^2-(b^2-4ac)}{4a^2}$

$= \frac{b^2-b^2+4ac}{4a^2}$

$= \frac{4ac}{4a^2}$

$= \frac{c}{a}$

3.4.2 Analytic Method

If α and β are the roots of the quadratic

equation $ax^2 + bx + c = 0$

Then $x = \alpha \; or \; x = \beta$.

$x - \alpha = 0 \; or \; x - \beta = 0$

$(x - \alpha)(x - \beta) = 0$

$x(x - \beta) - \alpha(x - \beta) = 0$ Open the bracket

$x^2 - \beta x - \alpha x + \alpha\beta = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Since $ax^2 + bx + c = 0$

Then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$(\boldsymbol{\alpha} + \boldsymbol{\beta}) = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$, $\boldsymbol{\alpha\beta} = \frac{\boldsymbol{c}}{\boldsymbol{a}}$

3.5 Method of Solutions for Quadratic Equation

Quadratic equations can be solved using the following methods

1. Factorization method.
2. Completing the square method
3. Formula method.
4. Graphical method.

Example 1: From the quadratic equation $6x^2 + 7x + 2 = 0$, Solve for x in the quadratic equation below using

a. Factorization method.
b. Completing the square method.
c. Formula method.

Solution:

3.5.1 Factorization Method

$6x^2 + x - 1 = 0$

Look for two values so that when you add or subtract them the result is +1 and when you multiply them the result is -6

$6x^2 + 3x - 2x - 1 = 0$

$3x(2x + 1) - 1(2x + 1) = 0$

$(3x - 1)(2x + 1) = 0$

$2x + 1 = 0 \; or \; 3x - 1 = 0$

$2x = -1 \; or \; 3x = 1$

$x = \frac{-1}{2} \; or \; x = \frac{-1}{3}$

3.5.2 Completing the square method

$6x^2 + x - 1 = 0$

$6x^2 + x = 1$ Divide both sides by 6

$$\frac{6x^2}{6} + \frac{x}{6} = \frac{1}{6}$$

$$x^2 + \frac{x}{6} = \frac{1}{6}$$

Add the square of half the coefficient of x to both sides. $\left(\frac{1}{12}\right)^2$

$$x^2 + \frac{x}{6} + \left(\frac{1}{12}\right)^2 = \frac{1}{6} + \left(\frac{1}{12}\right)^2$$

$$\left(x + \frac{1}{12}\right)^2 = \frac{1}{6} + \left(\frac{1}{12}\right)^2$$

$$\left(x + \frac{1}{12}\right)^2 = \frac{1}{6} + \frac{1}{144}$$

$$\left(x + \frac{1}{12}\right)^2 = \frac{24+1}{144}$$

$$\left(x + \frac{1}{12}\right)^2 = \frac{25}{144}$$

$$\left(x + \frac{7}{12}\right) = \pm\sqrt{\frac{25}{144}}$$

$$x = -\frac{1}{12} \pm \frac{5}{12}$$

$$x = \frac{-1\pm5}{12}$$

$$x = \frac{-1+5}{12} \ or \ \frac{-1-5}{12}$$

$$x = \frac{4}{12} \ or \ \frac{-6}{12}$$

$x = -\frac{1}{2}\ or\ \frac{1}{3}$

3.5.3 Formula Method:

Recall that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = 0$

$6x^2 + x - 1 = 0$

Comparing both equation $a = 6$, $b = 1$, $c = -1$.

Substitute a , $b\ and\ c$ into the Algebraic formula

$x = \frac{-1 \pm \sqrt{1^2 - (4 \times 6 \times -1)}}{2 \times 6}$

$x = \frac{-1 \pm \sqrt{1+24}}{12}$

$x = \frac{-1 \pm \sqrt{25}}{12}$

$x = \frac{-1 \pm 5}{12}$

$x = \frac{-1+5}{12}\ or\ \frac{-1-5}{12}$

$x = \frac{-6}{12}\ or\ \frac{4}{12}$

$x = -\frac{1}{2}\ or\ \frac{1}{3}$

EXAMPLE 2: If α and β are the roots of the quadratic equation $4x^2 + x - 3 = 0$ solve for

a) $\alpha + \beta$

b) $\alpha\beta$

c) $\alpha^2 + \beta^2$

d) $\frac{1}{\alpha} + \frac{1}{\beta}$

e) $\alpha^2 - \beta^2$

f) $\alpha^3 - \beta^3$ Hint: $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$

g) $\alpha^3 + \beta^3$ Hint: $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^3 b + 3\alpha\beta^3 + \beta^3$

Solution:

$a = 4, b = 1, c = -3$

a. Sum of the roots $\alpha + \beta = -\frac{b}{a} = -\frac{1}{4}$

b. Product of the roots $\alpha\beta = \frac{c}{a} = \frac{-3}{4}$

c. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{-1}{4}\right)^2 - 2\left(\frac{-3}{4}\right) = \frac{1}{16} + \frac{6}{4} = \frac{1+4(6)}{16} = \frac{1+24}{16} = \frac{25}{16}$

d. $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-1}{4} \div \frac{-3}{4} = \frac{-1}{4} \times \frac{4}{-3} = \frac{1}{3}$

e. $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

$\alpha - \beta$ *can't be solved directly. Therefore we solve for*

$(\alpha - \beta)^2 = (\alpha - \beta)(\alpha - \beta)$

$= \alpha^2 + \beta^2 - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta$

$= (\alpha + \beta)^2 - 4\alpha\beta$

$(\alpha - \beta)^2 = \left(\frac{-1}{4}\right)^2 - 4\left(\frac{-3}{4}\right)$

$= \frac{1}{16} + \frac{12}{4} = \frac{1+48}{16}$

$(\alpha - \beta)^2 = \frac{49}{16}$ *Therefore* $\alpha - \beta = \frac{7}{4}$

Hence, $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = \frac{-1}{4} \times \frac{7}{4} = -\frac{7}{16}$

f. $\alpha^3 - \beta^3$ Hint: $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) = (\alpha - \beta)[(\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta]$

$(\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta] = \frac{7}{4}\left[\left(\frac{-1}{4}\right)^2 - \left(\frac{-3}{4}\right)\right]$

$= \frac{7}{4}\left[\frac{1}{16} + \frac{3}{4}\right] = \frac{7}{4}\left[\frac{1+4(3)}{16}\right] = \frac{7}{4}\left[\frac{13}{16}\right] = \frac{91}{64}$

EXAMPLE 3: The quadratic equation $4x^2 + x - 3 = 0$ has its roots. Solve the equation whose roots are α^2 *and* β^2

Solution:

$a = 4, b = 1$ *and* $c = -3$

$\alpha + \beta = -\frac{b}{a} = -\frac{1}{4}$ and $\alpha\beta = \frac{c}{a} = \frac{-3}{4}$

Sum of the roots

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{-1}{4}\right)^2 - 2\left(\frac{-3}{4}\right) = \frac{1}{16} +$

$\frac{6}{4} = \frac{1+4(6)}{16} = \frac{1+24}{16} = \frac{25}{16}$

Product of roots

$\alpha^2 \times \beta^2 = (\alpha\beta)^2 = \left(\frac{-3}{4}\right)^2 = \frac{9}{16}$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - (\alpha^2 + \beta^2)x + (\alpha\beta)^2 = 0$

$x^2 - \frac{25}{16}x + \frac{9}{16} = 0$

$16x^2 - 25x + 9 = 0$

Example 4: α and β are the roots of the quadratic equation $px^2 + qx + r = 0$ where $p,\ q$ and r are constants such that $\alpha = 3\beta$. Show that $3q^2 = 16pr$

Solution:

$\alpha + \beta = -\frac{q}{p}$ (1)

$\alpha\beta = \frac{r}{p}$ (2)

$3\beta^2 = \frac{r}{p}$

$\beta^2 = \frac{r}{3p}$

$3\beta + \beta = -\frac{q}{p}$ so that $4\beta = -\frac{q}{p}$ which implies that

$\beta = -\frac{q}{4p}$

$\beta^2 = \frac{q^2}{16p^2}$

$\therefore \beta^2 = \frac{q^2}{16p^2} = \frac{r}{3p}$

$3pq^2 = 16p^2r$

$3q^2 = 16pr$

Assessment:

1. α and β are the roots of the quadratic equation $3x^2 - 5x + 2 = 0$ solve for
   a. $\frac{\beta}{\alpha} + \frac{\alpha}{\beta}$
   b. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
   c. $\alpha^3 + \beta^3$
2. α and β are the roots of the quadratic equation $2x^2 - 5x + 3 = 0$ solve for the quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$
3. Solve for x in the quadratic equation (i) $4x^2 + 8x + 3 = 0$ (ii) $x^2 + 2x + 1 = 0$ (iii) $5x^2 + 4x +$

$1 = 0$ using, Factorization method, Completing the square method and Formula method.

4. Solve for the value of b in the equation $27x^2 + bx + 8 = 0$ if one of the root is the square of the other.
5. Solve for the possible values of a if the equation $x^2 + (a + 1)x + a^2 - 5 = 0$ has real roots.
   The equation $x^2 - 3x + 1 = p(x - 3)$ has equal roots, what are the possible values of p
6. Solve for x in the quadratic equation $4x^2 + 8x + 3 = 0$ using
   a. Factorization method.
   b. Completing the square method.
   c. Formula method.
7. If one root of the equation $x^2 + bx + c = 0$ is k times the other where b, c and k are constants.
   Show that $c = k\left(\frac{b}{k+1}\right)^2$

CHAPTER FOUR

POLYNOMIAL

AIM: The overall aim of this lecture is to solve problems of a Polynomial Function.

OBJECTIVES: At the end of this lecture students should be able to:

1. Evaluate a Polynomial Expression.
2. Use the Remainders and Factor theorem to factorize a Polynomial.
3. Solve for unknown constants in a Polynomial expression
4. Factorize a quartic Polynomial

4.1 Definition of Polynomial

A polynomial as a mathematical expression which is the sum of terms, with each term being the product of a constant and a non-negative or zero power of a variable

$$a_0 + a_1x + a_2x^2 + a_3x^3 \dots\dots\dots a_nx^n$$

The highest power of the variable (n) is called the degree of the polynomial.

a_0 is a constant a_i $(i = 1,2, \dots \dots . n)$ Coefficients of the polynomial, $a_n > 0$ is the leading coefficient

A function whose value is given by the polynomial is called a polynomial function

An equation obtained when the polynomial equals zero is called a polynomial equation.

Polynomial Equation

$5x^3 - 3x^2 + 4x + 7 = 0$

Polynomial function

$F(x) = 5x^3 - 3x^2 + 4x + 7$

4.2 Classification of Polynomials According to Degrees.

DEGREE	NAME	EXAMPLE
1	LINEAR	$4x + 7$
2	QUADRATIC	$3x^2 + 4x + 7$
3	CUBIC	$5x^3 - 3x^2 + 4x + 7$
4	QUARTIC	$8x^4 + 5x^3 - 3x^2 + 4x + 7$

5	QUINTIC	$10x^5 + 8x^4 + 5x^3 - 3x^2 + 4x + 7$

4.3 Equality of Polynomial

When two polynomials F(x) and G(x) are equal

$F(x) = a_0 + a_1x + a_2x^2 \ldots\ldots\ldots a_nx^n$

$G(x) = b_0 + b_1x + b_2x^2 \ldots\ldots\ldots b_nx^n$

$a_0 = b_0\ ,\ a_1 = b_1,\ a_2 = b_2,\ a_n = b_n$

4.4 Addition and Subtraction of Polynomials

Suppose,

$F(x) = a_0 + a_1x + a_2x^2 \ldots\ldots\ldots a_nx^n$

$G(x) = b_0 + b_1x + b_2x^2 \ldots\ldots\ldots b_nx^n$

$F(x) + G(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots (a_n + b_n)x^n$

$F(x) - G(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots (a_n - b_n)x^n$

EXAMPLE 1:

Given that $f(x) = 9x^3 - 4x^2 + 3x + 7$, $G(x) = 6x^2 + 2x - 4$ and $H(x) = 5x^3 + 3x - 2$, Find

(a) $F(x) + G(x)$

(b) $F(x) + 2G(x) - 3H(x)$

Solution.

(a) $F(x) + G(x) = (9x^3 - 4x^2 + 3x + 7) + (6x^2 + 2x - 4)$

$= 9x^3 - 4x^2 + 3x + 7 + 6x^2 + 2x - 4$

$= 9x^3 - 4x^2 + 6x^2 + 3x + 2x + 7 - 4$

$= 9x^3 + 2x^2 + 5x + 3$

(b) $F(x) + 2G(x) - 3H(x) = (9x^3 - 4x^2 + 3x + 7) + 2(6x^2 + 2x - 4) - 3(5x^3 + 3x - 2)$

$= 9x^3 - 4x^2 + 3x + 7 + 12x^2 + 4x - 8 - 15x^3 - 9x + 6$

$= 9x^3 - 15x^3 - 4x^2 + 12x^2 + 3x + 4x - 9x + 7 - 8 + 6$

$= -6x^3 + 8x^2 - 2x + 5$

EXAMPLE 2:

If $F(x) = 2x^3 + 4x^2 - 5x + 5$, find

(a) F(-1)

(b) F(1)

Solution.

(a) $F(-1) = 2(-1^3) + 4(-1^2) - 5(-1) + 5$

$= 2(-1) + 4(1) - 5(-1) + 5$

$= -2 + 4 + 5 + 5 = 12$

(b) $F(1) = 2(1^3) + 4(1^2) - 5(1) + 5$

$= 2(1) + 4(1) - 5(1) + 5$

$= 2 + 4 - 5 + 5 = 6$

4.5 Products of Polynomial

This is when a polynomial of degree m is multiplied by a polynomial of degree n.

Example 3:

Given that $F(x) = 3x^2 + 4x + 2$ and $G(x) = 4x^3 - 2x^2 + 5x + 6$, find $F(x).G(x)$

Solution

Method 1:

$F(x).G(x) = (3x^2 + 4x + 2)(4x^3 - 2x^2 + 5x + 6)$

$= 3x^2(4x^3 - 2x^2 + 5x + 6) + 4x(4x^3 - 2x^2 + 5x + 6) + 2(4x^3 - 2x^2 + 5x + 6)$

$= 12x^5 - 6x^4 + 15x^3 + 18x^2 + 16x^4 - 8x^3 + 20x^2 + 24x + 8x^3 - 4x^2 + 10x + 12$

$= 12x^5 - 6x^4 + 16x^4 + 15x^3 - 8x^3 + 8x^3 + 18x^2 + 20x^2 - 4x^2 + 24x + 10x + 12$

$= 12x^5 + 10x^4 + 15x^3 + 34x^2 + 34x + 12$

Method 2:

$$\begin{array}{r}
4x^3 - 2x^2 + 5x + 6 \\
3x^2 + 4x + 2 \\
\hline
8x^3 - 4x^2 + 10x + 12 \\
16x^4 - 8x^3 + 20x^2 + 24x \quad \\
12x^5 - 6x^4 + 15x^3 + 18x^2 \qquad\qquad \\
\hline
12x^5 + 10x^4 + 15x^3 + 34x^2 + 34x + 12
\end{array}$$

Exercise: $F(x) = 3x^2 + 3x + 4$; $G(x) = 5x^2 - 6x + 8$ and $H(x) = 9x^3 - 3x^2 + 5x + 6$. Solve for:

I. $F(x) + G(x) + H(x)$

II. $3F(x) - G(x) + 2H(x)$

III. $4G(x) + 2H(x) - 2F(x)$

4.6 Division of Polynomials

Suppose the polynomial $5x^2 + 3x + 2$ is divided by the polynomial $x + 4$, the polynomial $5x^2 + 3x + 2$ is the dividend, $x + 4$ is the divisor, the result is the quotient and the left after the division is the remainder.

$\boldsymbol{F(x) = D(x).Q(x) + R}$

$F(x)$ = Dividend, $D(x)$ = Divisor, $Q(x)$ = Quotient and R = Remainder.

EXAMPLE 4:

Find the quotient and the remainder when $8x^2 + 6x + 5$ is divided by $2x - 2$

Solution.

$$
\begin{array}{r|l}
 & 4x + 7 \\
\hline
2x - 2 & 8x^2 + 6x + 5 \\
 & \underline{8x^2 - 8x} \\
 & 14x + 5 \\
 & \underline{14x - 14} \\
 & 19
\end{array}
$$

The quotient is $4x + 7$ while the remainder is 19

4.7 Remainders Theorem

If the polynomial $F(x)$ is divided by $x - a$, the remainder is $F(x)$.

Proof

$F(x) = (x - a)Q(x) + R$

$Q(x)$ = Quotient, $(x - a)$ = Divisor, R = remainder.

$F(a) = (a - a)q(a) + R$

$F(a) = R$

Example 5: Find the remainder when $2x^2 - 5x + 1$ is divided by $x - 3$.

Solution

Let $F(x) = 2x^2 - 5x + 1$ and R = remainder when $F(x)$ is divided by $x - 3$.

Then $x - 3 = 0$ hence $x = 3$

$R = F(3)$

$F(3) = 2(3^2) - 5(3) + 1 = 18 - 15 + 1 = 4$

4.8 Factor Theorem

If $f(x)$ is divided by $x - a$ the remainder is 0

A special case arises however when $f(x)$ leaves no remainder when it is divided by $x - a$, we say that $x - a$ is a factor of $f(x)$.

Furthermore, if $f(x) = 0$, then $x - a$ is a factor of $f(x)$.

This modified theorem is called Factors theorem

Example 6:

If $x + 2$ is a factor of polynomial $x^3 + x^2 + kx + 12k^2$, find the possible values of the constant k

Solution.

Let $f(x) = x^3 + x^2 + kx + 12k^2$

If $x + 2$ is a factor of $f(x)$, then $f(-2) = 0$,

$f(-2) = -2^3 + (-2)^2 + k(-2) + 12k^2 = 0$

$= -8 + 4 - 2k + 12k^2 = 0$

$12k^2 - 2k - 4 = 0$

$6k^2 - k - 2 = 0$

$6k^2 - 4k + 3k - 2 = 0$

$2k(3k - 2) + 1(3k - 2) = 0$

$(3k - 2)(2k + 1) = 0$

$k = \frac{-1}{2} \;\; or \; k = \frac{2}{3}$

Example7: Find the values of the constant a and b such that the remainder of $f(x)=6x^3+ax^2-bx-2$ is 6 when f(x) is divided by (x-1) and 0 when it is divided by (3x-2). Factorize f(x) completely and state its zeros.

Solution:

$f(x) = 6x^3 + ax^2 - bx - 2$

for (x-1); $f(1) = 6(1)^3 + a(1)^2 - b(1) - 2 = 6$

$$6 + a - b - 2 = 6$$

$$a - b = 6 - 6 + 2$$

$$a - b = 2 \ldots\ldots\ldots\ldots (1)$$

for (3x-2); $f\left(\frac{2}{3}\right) = 6\left(\frac{2}{3}\right)^3 + a\left(\frac{2}{3}\right)^2 - b\left(\frac{2}{3}\right) - 2 = 0$

$$6\left(\frac{8}{27}\right) + a\left(\frac{4}{9}\right) - b\left(\frac{2}{3}\right) - 2 = 0$$

$$\left(\frac{16}{9}\right) + \left(\frac{4a}{9}\right) - \left(\frac{2b}{3}\right) = 2$$

$$\frac{16+4a-6b}{9} = 2$$

$$16 + 4a - 6b = 18$$

$$4a - 6b = 18 - 16$$

$$4a - 6b = 2$$

$$2a - 3b = 1 \ldots\ldots\ldots\ldots (2)$$

$$a - b = 2 \ldots\ldots\ldots\ldots (1) \ *2$$

$2a - 3b = 1$………..(2) *1

$2a - 2b = 4$…………. (1)*2

$2a - 3b = 1$………..(2) *1

(2) - (1); $-b = -3$

$b = 3$

from (1) $a - b = 2$

$a - 3 = 2$

$a = 2 + 3$

$a = 5$

$a = 5, b = 3$

4.9 Roots of Cubic Equation

Cubic equation takes the form

$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0)$ …………(1)

Divide equation (1) through by a, we have

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ ……………..(2)

Given that α, β, γ are the roots of the equation

$(x - \alpha)(x - \beta)(x - \gamma) = 0$ ………………(3)

Where $x = \alpha, x = \beta, x = \gamma$

Expand equation (3)

$(x - \alpha)[x(x - \gamma) - \beta(x - \gamma)]$

$= (x - \alpha)[x^2 - \beta x - \gamma x + \beta\gamma]$

$= x(x^2 - \beta x - \gamma x + \beta\gamma) - \alpha(x^2 - \beta x - \gamma x + \beta\gamma)$

$= x^3 - \gamma x^2 - \beta x^2 + \beta\gamma x - \alpha x^2 + \alpha\gamma x + \alpha\beta x - \alpha\beta\gamma$

$= x^3 - \gamma x^2 - \beta x^2 - \alpha x^2 + \alpha\gamma x + \alpha\beta x + \beta\gamma x - \alpha\beta\gamma$

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma$

……(4)

Comparing equation 4 and 2,

$\frac{b}{a} = -(\alpha + \beta + \gamma),\ \frac{c}{a} = (\alpha\beta + \alpha\gamma + \beta\gamma),\ \frac{d}{a} = -\alpha\beta\gamma$

Therefore, the sum and the products of the roots of a cubic polynomial is

$-\frac{b}{a} = (\alpha + \beta + \gamma)$ and $-\frac{d}{a} = \alpha\beta\gamma$

EXAMPLE 8:

What is the product and sum of the roots of the cubic polynomial $6x^3 + 5x^2 - 3x - 2$

Solution

$a = 6, b = 5, c = -3, d = -2$

Sum of the roots $= (\alpha + \beta + \gamma) = -\frac{b}{a} = -\frac{5}{6}$

Product of the roots $= \alpha\beta\gamma = -\frac{d}{a} = -\frac{-2}{6} = \frac{1}{3}$

Solving a Quartic Polynomial of Degree 4

EXAMPLE 9:
Factorize $F(x) = 2x^4 - 5x^3 - 15x^2 + 10x + 8$
Solution

$F(1) = 0$

$F(1) = 2(1^4) - 5(1^3) - 15(1^2) + 10(1) + 8$

$= 2(1) - 5(1) - 15(1) + 10(1) + 8$

$= 2 - 5 - 15 + 10 + 8 = 0$

$F(-2) = 2(-2^4) - 5(-2^3) - 15(-2^2) + 10(-2) + 8$

$= 2(16) - 5(-8) - 15(4) + 10(-2) + 8$

$= 32 + 40 - 60 - 20 + 8 = 0$

$x = 1$ and $x = 2$, therefore $x - 1 = 0, x - 2 = 0$

$(x - 1)(x + 2) = x^2 + 2x - x - 2$

$= x^2 + x - 2$

$$
\begin{array}{r|l}
 & 2x^2 - 7x - 4 \\
\hline
x^2 + x - 2 & 2x^4 - 5x^3 - 15x^2 + 10x + 8 \\
 & 2x^4 + 2x^3 - 4x^2 \\
 & \quad -7x^3 - 11x^2 + 10x \\
 & \quad -7x^3 - 7x^2 + 14x \\
 & \qquad -4x^2 - 4x + 8 \\
 & \qquad -4x^2 - 4x + 8 \\
 & \qquad \cdot \quad \cdot \quad \cdot
\end{array}
$$

4.10 Graphs of a Polynomial Function

4.10 Graphs of a Polynomial Function

The graph of a polynomial of degree one is a straight line

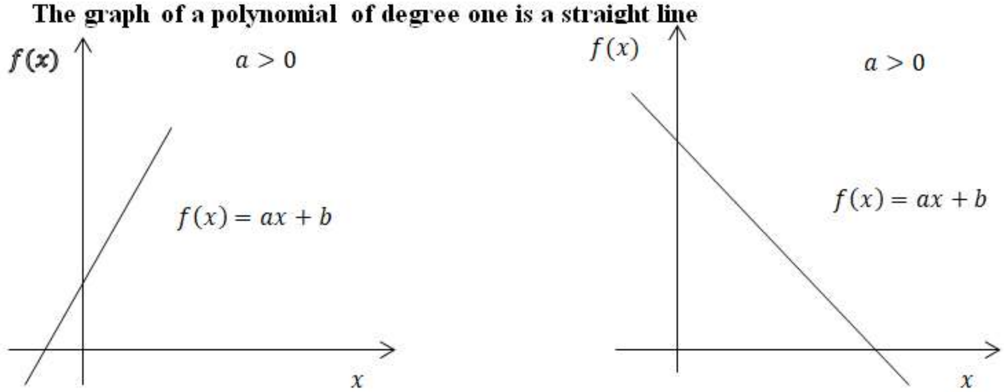


Figure 1.1 Graph of a polynomial of degree one

The graph of polynomial of degree 2 is called a parabola

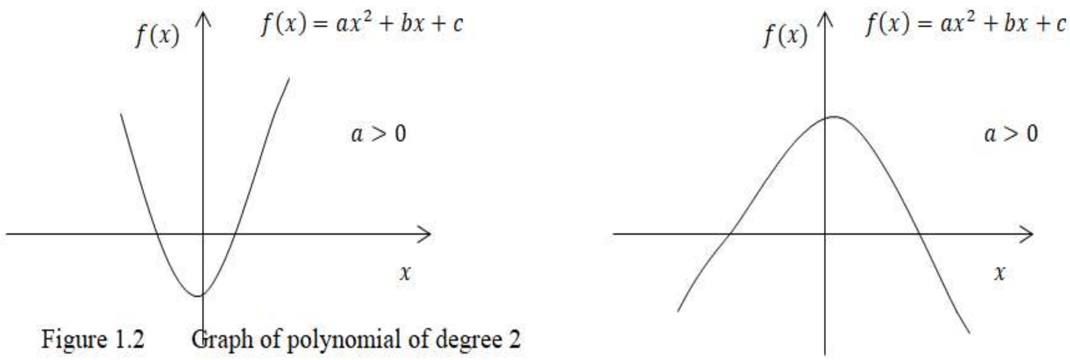


Figure 1.2 Graph of polynomial of degree 2

The graph of a polynomial of degree three is called a cubical parabola

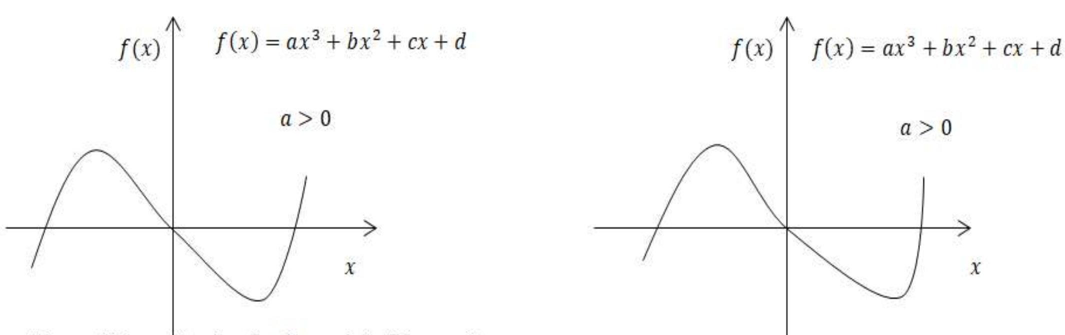


Figure 1.3 Graph of polynomial of degree 3

Assessment:

1. Given that $cx^4 + dx^2 + 7x + 2d =$ is divided by $x - 1$ the remainder is -2, and if by $x + 2$ the remainder is 58. Solve for the values of c and d.
2. The polynomial $2x^3 + qx^2 + rx + 2$ has a factor $x - 1$ and leaves a remainder of 12 when divided by $x - 2$. Find the constants q and r and the factors of the polynomial.
3. The remainder when $f(x) = x^3 + ax^2 + bx + c$ is divided by $x - 1$, $x + 2$ and $x - 2$ are respectively 2 , -1 and 15, solve for the quotient and remainder when $f(x)$ is divided by $x + 1$.
4. What is the remainder when $5x^3 + 4x^2 + 3x - 12$ is divided by $x - 4$
5. Factorize the following quartic equations $f(x) = 2x^4 - 5x^3 - 15x^2 + 10x + 8$
6. What is the sum and product of the cubic polynomial expression $2x^3 - 9x^2 + 13x - 6$

CHAPTER FIVE
PARTIAL FRACTION

5.1 Definition of Partial Fraction

To have a partial fraction, the degree of the numerator should be less than the degree of the denominator. We cannot discuss partial fraction without discussing a rational fraction

A rational fraction $A(x)$ is a fraction of two polynomials $B(x)$ and $C(x)$ I.E

$$A(x) = \frac{B(x)}{C(x)}$$

This fraction could be broken down into smaller fraction and whole number or not broken down at all. The components obtained from the rational fraction by partial fractionalization is called partial fraction

Examples are

$$\frac{x}{14x^2+x} \text{ or } \frac{2+x}{2-x}$$

In partial fraction, we take the LCM of the individual denominators

For example $\frac{1}{x-3} - \frac{3}{x-1}$ has the LCM $(x-3)(x-1)$

Therefore,

$$\frac{1}{x-3} - \frac{3}{x-1} = \frac{1(x-1)-3(x-3)}{(x-3)(x-1)} = \frac{x-1-3x+9}{(x-3)(x-1)} = \frac{8-2x}{(x-3)(x-1)}$$

Therefore,

$$\frac{1}{x-3} - \frac{3}{x-1} = \frac{8-2x}{(x-3)(x-1)}$$

A polynomial function whose degree of the numerator is greater than the degree of the numerator i.e $B^m(x) \geq C^n(x)$ where m & n are the degree or order of the polynomial $B(x)$ and $C(x)$ will have the expression

$$\frac{B^m(x)}{C^n(x)} = D^r(x) + \frac{E^q(x)}{C^n(x)}$$

Where

$$B^m(x) = C^n(x)D^r(x) + E^q(x)$$

$D^r(x) = Remainder, B^m(x) = Dividend, C^n(x) = Divisor, E^q(x) = quotient$

5.2 Different Approach in solving Partial Fraction Problem

Approach 1

$$\frac{B^m(x)}{C^n(x)}$$

$B^m(x)$ is a linear function and $C^n(x)$ is a quadratic function

Example 1: Solve the partial fraction $\frac{5x-4}{x^2-x-2}$

Solution:

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

$$\frac{5x-4}{x^2-x-2} = \frac{A(x+1)}{(x-2)} + \frac{B(x-2)}{(x+1)} = \frac{A(x-4)+B(x-2)}{(x-2)(x+1)}$$

$\frac{5x-4}{x^2-x-2} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$ This means that,

$$5x - 4 = A(x+1) + B(x-2)$$

When $x = -1$, Then $5(-1) - 4 = A(-1+1) + B(-1-2)$

$-9 = -3B$ Then $B = 3$

When $x = 2$, Then $5(2) - 4 = A(2+1) + B(2-2)$

$6 = 3A$ Then $A = 2$

$$\frac{5x-4}{x^2-x-2} = \frac{2}{(x-2)} + \frac{3}{(x+1)}$$

Example 2: Express $\frac{12x^2+37x+26}{2x^2+5x+3}$ in partial fraction.

Solution:

Let us first divide the polynomial and get the quotient and the remainder.

$$\begin{array}{r} 6 \\ 2x^2+5x+3\overline{\smash{)}12x^2+37x+26} \\ -12x^2+30x+18 \\ \hline 7x+8 \\ \hline \end{array}$$

Factorize $2x^2 + 5x + 3 = 2x^2 + 3x + 2x + 3 =$

$2x(x + 1) + 3(x + 1) = (2x + 3)(x + 1)$

$\frac{12x^2+37x+26}{2x^2+5x+3} = 6 + \frac{7x+8}{(2x+3)(x+1)}$ Where $\frac{7x+8}{(2x+3)(x+1)} =$

$\frac{A}{(2x+3)} + \frac{B}{(x+1)}$

$7x + 8 = A(x + 1) + B(2x + 3)$

When $x = -1, 7(-1) + 8 = A(0) + B(-2 + 3)$

$1 = B$. This implies that $B = 1$

When $x = -\frac{3}{2}, 7\left(-\frac{3}{2}\right) + 8 = A\left(-\frac{3}{2} + 1\right) +$

$B\left(2\left(-\frac{3}{2}\right) + 3\right)$

$-\frac{21}{2} + 8 = A\left(\frac{-3+2}{2}\right)$

$\frac{-21+16}{2} = \frac{-A}{2}$

This implies that $A = 5$

$\frac{12x^2+37x+26}{2x^2+5x+3} = 6 + \frac{7x+8}{(2x+3)(x+1)} = 6 + \frac{5}{(2x+3)} + \frac{1}{(x+1)}$

Approach 2

In this approach, the denominator of the fraction is expressed as a quadratic function that cannot be reduced

Example 3: Express $\frac{10x^2-2x+1}{(x+1)(2x^2+3x-11)}$ in partial fraction.

Solution:

$$\frac{10x^2-2x+1}{(x+1)(2x^2+3x-11)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+3x-11}$$

$$\frac{10x^2-2x+1}{(x+1)(2x^2+3x-11)} = \frac{A(2x^2+3x-11)+(Bx+C)(x+1)}{x+1(2x^2+3x-11)}$$

$$10x^2 - 2x + 1 = A(2x^2 + 3x - 11) + (Bx + C)(x + 1)$$

$$10x^2 - 2x + 1 = 2Ax^2 + 3Ax - 11A + Bx^2 + Bx + Cx + C$$

$10 = 2A + B$……………… (1)

$-2 = 3A + B + C$ ……….. (2)

$1 = -11A + 1C$ ………….. (3)

From (1) $B = 10 - 2A$ and from (3) $C = 11A + 1$

Substitute B and C into equation (2)

$$-2 = 3A + 10 - 2A + 11A + 1$$

$-2 = 12A + 11$

$-12A = 13$

$A = \frac{-13}{12}$

From (1) $B = B = 10 - 2A = 10 - 2\left(\frac{-13}{12}\right) = 10 + \frac{13}{6} = \frac{73}{4}$

$B = \frac{73}{4}$

From (3) $C = 1 + 11A = 1 + 11\left(\frac{-13}{12}\right) = \frac{12-143}{12} = \frac{-131}{12}$

$C = \frac{-131}{12}$

$$\frac{10x^2-2x+1}{(x+1)(2x^2+3x-11)} = \frac{-13}{12(x+1)} + \left[\frac{146x-131}{12(12x^2+3x-11)}\right]$$

Approach 3

This approach involves rational polynomial function with the denominator having a repeated simple factor.

Example 4: Express $\frac{-10x+5}{(2x+1)^2}$ as a partial fraction

Solution

$$\frac{-10x+5}{(2x+1)^2} = \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2}$$

$$-10x + 5 = A(2x+1)^2 + B(2x+1)$$

$-10x + 5 = A(2x + 1)(2x + 1) + B(2x + 1)$

$-10x + 5 = 2Ax(2x + 1) + A(2x + 1) + B(2x + 1)$

$-10x + 5 = 4Ax^2 + 2Ax + 2Ax + A + 2Bx + B$

$-10x + 5 = 4Ax^2 + 4Ax + 2Bx + A + B$

We form two simultaneous equations from the equation above

$4A + 2B = -10$ ……………………………………(1)

$A + B = 5$ ………………………………………………(2)

From equation (2), $B = 5 - A$

Substitute B into equation (1)

$4A + 2(5 - A) = -10$

$4A + 10 - 2A = -10$

$2A = -10 - 10$

$2A = -20$

$A = -10$

Substitute A into B

$B = 5 - (-10) = 5 + 10 = 15$

$\frac{-10x+5}{(2x+1)^2} = \frac{-10}{(2x+1)} + \frac{15}{(2x+1)^2}$

Assessment

Solve

1. Express $\frac{8x-28}{x^2-6x+8}$ in partial fraction.
2. Express $\frac{7x-1}{x^2+x-2}$ in partial fraction.
3. Express $\frac{3x+4}{x^2+3x+2}$ in partial fraction.
4. Express $\frac{6x^2+20x+10}{2x^2+5x+2}$ in partial fraction.
5. Express $\frac{5x^3+x^2+11x+1}{5x^2-4x-1}$ in partial fraction.
6. Express $\frac{x^2+3x-10}{x^2-2x-3}$ in partial fraction.
7. Express $\frac{3x^3-x^2-13x-13}{x^2-x-6}$ in partial fraction.
8. Express $\frac{5x^2+3x-1}{(x+1)(2x^2+3x-11)}$ in partial fraction.
9. **Express** $\frac{5x+1}{(3x+1)^3}$ in partial fraction

CHAPTER SIX

ARITHMETIC PROGRESSION

AIM: The overall aim of this lecture is to solve problems involving Arithmetic Progression.

OBJECTIVES: at the end of this lecture students should be able to:

1. Define a sequence and a series and give examples
2. Explain the meaning of an arithmetic progression (A.P)
3. Solve for the first term, common difference and the nth term of an (A.P)
4. Solve for the sum of the nth term of an (A.P)

6.1 Sequence and Series

SEQUENCE: A set of objects or numbers or events presented in an orderly manner.

TYPES OF SEQUENCE: Finite and Infinite sequence

FINITE SEQUENCE: When the number of terms in a sequence is limited. An Example is 1, 2, 3, 4.

INFINITE SEQUENCE: When the number of terms in a sequence is unlimited. An Example is 1, 2, 3, 4, …….

SERIES: This is the summation or addition of the terms of a sequence.

TYPES OF SERIES: Finite and Infinite series

FINITE SERIES: When the number of terms in a series is limited. An Example is 1+2+3+4

INFINITE SERIES: When the number of terms in a series is unlimited. An Example is 1+2+3+4+ …….

EXAMPLE 1: Given that $U_n = \frac{1}{n^2}, where \ (n = 1,2,3, \ldots\ldots)$ write out the first four terms of the sequence.

Solution:

$U_n = \frac{1}{n^2}$

$U_1 = \frac{1}{1^2} = \frac{1}{1} = 1$

$U_2 = \frac{1}{2^2} = \frac{1}{4}$

$U_3 = \frac{1}{3^2} = \frac{1}{9}$

$U_4 = \frac{1}{4^2} = \frac{1}{16}$

Hence the first four terms are $\mathbf{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}}$

APPLICATION OF SEQUENCE AND SERIES:

Sequence and series is applied to medicine (calculating the amount of drugs in the body etc.), engineering and science.

6.2 Arithmetic Progression (A.P)

This is an arithmetic sequence where each term differs from the preceding term by a constant amount. An example is 1, 3, 5, 7, ……..

6.3 Nth Term of an (A.P)

The nth term of an arithmetic progression is also called the last term of the arithmetic progression

$a,\ a+d, a+2d, a+3d, \dots\dots..a+(n-1)d$

$a =$ First term, $d =$ Common difference (The difference between the next term and the preceding term), $n =$ Number of terms and $T_n =$ nth term or last term of the A.P

Nth term of an A.P is $T_n = a + (n-1)d$

EXAMPLE 2: Given the A.P 3, 6, 9, …… Solve,

a) The first term

b) The common difference

c) The 9th term

Solution:

a) $a =$ First term = 3

b) $d =$ Common difference = $6 - 3 = 3$

c) $T_n = a + (n-1)d$

$a = 3, d = 3, n = 9$

$T_9 = 3 + (9-1)3$

$T_9 = 3 + (8)3$

$T_9 = 3 + 24$

$T_9 = 27$

EXAMPLE 3: The 9th term of an A.P is 50 and the first term is 2, Solve for the common difference.

Solution:

$a = 2, = ?, n = 9, T_9 = 50$

$T_n = a + (n-1)d$

$T_{11} = 2 + (9-1)d = 50$

$2 + (9-1)d = 50$

$2 + 8d = 50$ (Collect like terms)

$8d = 50 - 2$

$8d = 48$ (Divide both sides by 8 to make d the subject of the formula)

$d = 6$ This is the common difference.

EXAMPLE 4: Solve the 19th term of the A.P

$\frac{5}{7}, \frac{8}{7}, \frac{11}{7}, \ldots$

Solution:

$a = \frac{5}{7}$, $d = \frac{8}{7} - \frac{5}{7} = \frac{8-5}{7} = \frac{3}{7}$, $n = 19, T_{19} = ?$

$T_n = a + (n-1)d$

$T_{19} = \frac{5}{7} + (19-1)\frac{3}{7}$

$T_{19} = \frac{5}{7} + (18)\frac{3}{7}$

$T_{19} = \frac{5}{7} + \frac{54}{7} = \frac{59}{7}$

$T_{19} = 8\frac{3}{7}$

EXAMPLE 5: The 8th term of an A.P is 36 and the 13th term is 76. Solve for the

a) First term
b) Common difference
c) 20th term of the A.P

Solution:

$T_n = a + (n - 1)d$

$T_8 = a + 7d = 36$

$T_{13} = a + 12d = 76$

$a + 7d = 36$ ………………. (1)

$a + 12d = 76$ …………….. (2)

Subtract equation 1 from 2

$5d = 40$

Divide both sides by 5

$d = \frac{40}{5}$

$d = 8$

Substitute d into equation (1)

From equation 1, $a + 7d = 36$

$a + 7(6) = 36$

$a + 42 = 35$ (Collect like terms)

$a = 35 - 42$

$a = -7$

a) First term $a = -7$
b) Common difference $d = 8$
c) $T_{20} = -7 + (20 - 1)8$

$T_{20} = -7 + (19)8$

$T_{20} = -7 + 152$

$T_{20} = 145$

6.4 Sum of the nth term of a Sequence

Recall the sequence,

$a + (a + d) + (a + 2d) + \cdots \ldots\ldots + [\boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}]$

$= a + (a + d) + (a + 2d) + \cdots \ldots\ldots + \boldsymbol{T_n}$

$\sum_{i=1}^{n} T_i \quad = a + (a + d) + (a + 2d) +$

$\cdots \ldots\ldots + T_n$ ……... (1)

$\sum_{i=1}^{n} T_i \quad = T_n + (T_n - d) + (T_n - 2d) + \cdots + a$

……... (2) Add equation 1. and 2.

$2\sum_{i=1}^{n} T_i = (a + T_n) + (a + T_n) + (a + T_n) +$

$\cdots \ldots . (a + T_n)$

$2\sum_{i=1}^{n} T_i = n(a + T_n)$

Divide both sides by 2

$\sum_{i=1}^{n} U_i \quad = \frac{n}{2}(a + T_n)$

$S_n = \frac{n}{2}(a + T_n)$

Recall that $\boldsymbol{T_n} = \boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}$.

Substitute T_n into the above equation

$S_n = \frac{n}{2}[a + a + (n - 1)d]$

$$\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]}$$

EXAMPLE 6: Solve the sum of the first twenty terms of the A.P 3, 7, 11, 15, ………

Solution:

$a = 3, d = 7 - 3 = 4, n = 16, S_{20} = \ ?$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{20} = \frac{20}{2}[2 \times 3 + (20-1)4]$$

$$S_{20} = 10[6 + (19)4]$$

$$S_{20} = 10[6 + 76]$$

$$S_{20} = 10[82]$$

$$S_{20} = 820$$

EXAMPLE 6: The sum of the first three terms of an A.P is 24 and the sum of the first five terms is 70.

Solve

a) The first term
b) The common difference
c) The ninth term
d) The sum of the first ten terms of the A.P

Solution:

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_3 = \frac{3}{2}[2a + 2d] = 24$

$\frac{3}{2}[2a + 2d] = 24$ Cross multiplying

$3[2a + 2d] = 24 \times 2$ Divide both sides by 3

$2a + 2d = \frac{24\times 2}{3}$ Factor out 2

$2(a + d) = 16$ Divide both sides by 2

$a + d = \frac{16}{2}$

$a + d = 8$ …….. (1)

$S_5 = \frac{5}{2}[2a + 4d] = 70$

$\frac{5}{2}[2a + 4d] = 70$ Cross multiply

$5[2a + 4d] = 70 \times 2$ Divide both sides by 5

$[2a + 4d] = \frac{70\times 2}{5}$ Factor out 2

$2[a + 2d] = 28$ Divide both sides by 2

$[a + 2d] = \frac{28}{2}$

$a + 2d = 14$ …. (2)

Re-arranging equation (1) and (2)

$a + d \quad = 8$ .... (1)

$a + 2d = 14$ .... (2) Subtract equation (1) from (2)

$d = 6$ Substitute d into equation (1)

$a + 6 = 8$ Collect like terms

$a = 8 - 6$

$a = 2$

a) First term $a = 2$
b) Common difference $d = 6$
c) Recall that

$T_n = a + (n - 1)d$

$T_9 = 2 + (9 - 1)6$

$T_9 = 2 + 8(6)$

$T_9 = 2 + 48$

$T_9 = 50$

d) Recall that

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_{10} = \frac{10}{2}[2 \times 2 + (10 - 1)6]$

$S_{10} = 5[4 + 9(6)]$

$S_{10} = 5[4 + 54]$

$S_{10} = 5[58]$

$S_{10} = 290$

Assessment:

1. The Arithmetic progression is given as 1, 3, 5, … Solve for the first term, common difference and the 20th term.
2. The first term of an A.P is equal to twice the common difference d. Solve in terms of d, the 5th term of the A.P.
3. The Eleventh term of an A.P is 25 and its first term is -3. Solve for its common difference.
4. The first term of an A.P is equal to thrice the common difference. Solve for the 6th term of the A.P if the common difference is 8.
5. Solve for the 10th term of the A.P 2, 4, 6….
6. The blood pressure of some patients measured and recorded at a hospital form an arithmetic progression. If the blood pressure of the 4th patient reads 96mmHg and146mmHg for the 6th patient. Solve for the blood pressure of the 10th patient.

7. The first term of an A.P is -8. The ratio of the 7^{th} term to the 9^{th} term is 5:8. Solve for the common difference of the progression.
8. The third and the seventh term of an A.P are 27 and 91 respectively. Solve,

a) The first term

b) The common difference

c) The 22^{nd} term of the A.P

9. Solve the sum of the first 30 terms of the A.P 3, 5, 7, …….
10. The sum of the first 16 terms of an A.P is 240 and the sum of the next 4 terms is 220. Solve the

a) First term

b) Common difference

c) 20^{th} term of the progression

d) The sum of the first 20 terms.

CHAPTER SEVEN

GEOMETRIC PROGRESSION

AIM: The overall aim of this lecture is to solve problems involving Geometric Progression

OBJECTIVES: at the end of this lecture students should be able to:

1. Explain the meaning of a Geometric progression (G.P) and give examples
2. Solve for the first term, common difference and the nth term of a (G.P)

Solve for the sum of the nth term of a (G.P)

7.1 Geometric Progression (G.P)

A geometric progression (G.P) is a form of sequence which has a common ratio between any of the terms and its preceding term.

For example 2, 4, 8, 16, … has a first term of 2 and a common ratio of 2.

7.2 Nth term of a Geometric Progression (G.P)

$a,\ ar,\ ar^2,\ ar^3,\ \dots\dots\ \boldsymbol{ar^{n-1}}$

The nth term of a G.P is also called the last term usually given by $\boldsymbol{U_n = ar^{n-1}}$ where a is the first term, r is the common ratio ($r = \frac{U_2}{U_1}$) and n is the number of terms.

EXAMPLE 1: Solve the common ratio in the G.P $\log 3, \log 9, \log 81$.

Solution:

$$r = \frac{\log 9}{\log 3} = \frac{\log 3^2}{\log 3} = \frac{2\log 3}{\log 3} = 2$$

EXAMPLE 2: Solve the 5th term of a G.P 3, 9, 27,…..

Solution:

$$a = 3,\ r = \frac{9}{3} = 3,\ n = 5$$

$$U_n = ar^{n-1}$$

$$U_5 = 3(3^{5-1}) = 3(3^4) = 3 \times 81 = 243$$

$$\boldsymbol{U_5 = 243}$$

EXAMPLE 3: The fourth term of a G.P is 1. If its first term is 512, solve the common ratio.

Solution:

$a = 512,\ r = ?,\ n = 4,\ U_4 = 1$, recall that $U_n = ar^{n-1}$

$U_4 = 1 = 512r^{4-1}$

$1 = 512r^3$ Divide both sides by 512

$\frac{1}{512} = r^3$ Take the cube root of both sides

$\sqrt[3]{\frac{1}{512}} = \sqrt[3]{r^3}$

$r = \left(\frac{1}{8}\right)^{3\times\frac{1}{3}} = \frac{1}{8}$

Unknown *a* and *r* Leading to an equation.

EXAMPLE 4: The 4th and 6th term of a G.P are 48 and 432 respectively. Solve for the fifth term.

Solution:

$U_n = ar^{n-1}$

$U_4 = ar^3 = 48$ ………..(1)

$U_6 = ar^5 = 432$ ………(2)

$ar^3 = 48$ ………..(3)

$ar^5 = 432$ ………(4)

Divide eqn. (4) by eqn. (3)

$\frac{r^5}{r^3} = \frac{432}{48}$

$r^2 = 9$

Take the cube root of both sides

$\sqrt[2]{r^2} = \sqrt{9}$

$r = 3$

Substitute r into eqn. (1)

$a(3)^3 = 48$

$27a = 48$

Divide both sides by 27

$a = \frac{48}{27} = \frac{16}{9}$

$U_5 = \frac{16}{9}(3)^4$

$U_4 = \frac{16}{9}(81) = 16 \times 9 = 144$

Hence the fifth term is 144.

7.3 Sum of the nth term of a Geometric Progression (G.P)

$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$ ………….(1)

$rS_n = ar + ar^2 + ar^3 + \cdots ... + ar^{n-1} + ar^n$ ……(2)

Subtract eqn. (2) from eqn. (1)

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$S_n = \frac{a(r^n-1)}{r-1}$ When $r > 1$

$S_n = \frac{a(1-r^n)}{1-r}$ When $r < 1$

EXAMPLE 5: Solve the sum of the first four terms of the G.P 2, 6, 18, ……

Solution:

$a = 2,\ r = \frac{6}{2} = 3$ We use $S_n = \frac{a(r^n-1)}{r-1}$ when $r > 1$

$S_4 = \frac{2(3^4-1)}{3-1} = \frac{2(81-1)}{2} = \frac{2(80)}{2} = 80$

$\boldsymbol{S_4 = 80}$

EXAMPLE 6: If the 2nd and 5th terms of a G.P are 10 and -80 respectively, solve the sum of the first four terms.

Solution:

$U_n = ar^{n-1}$

$U_2 = ar = 10$ ...……..(1)

$U_5 = ar^4 = -80$ ……(2)

$ar = 10$ ...……………(1)

$ar^4 = -80$ …………..(2)

Divide eqn. (1) by eqn. (2)

$\frac{r^4}{r} = \frac{-80}{10}$

$r^3 = -8$

$r^3 = -2^3$

$r = -2$

Substitute r into eqn. (1)

$a \times (-2) = 10$

Divide both sides by 2

$a = \frac{10}{-2} = -5$

$a = -5,\ r = -2\ and\ r < 1$

$S_n = \frac{a(1-r^n)}{1-r}$ When $r < 1$

$S_4 = \frac{-5(1-(-2)^4)}{1-(-2)} = \frac{-5(1-16)}{1+2} = \frac{-5(-15)}{3} = 25$

The sum of the first four terms = 25

Assessment:

1. The common ratio of a G.P is 2. If the 5th term is greater than the first by 45. Solve the fifth term.
2. Solve for the 5th term of $\frac{8}{9},\ -\frac{4}{3},\ 2,\ \ldots..$
3. The 6th term of a G.P is $-\frac{2}{27}$ and the first term is 18. Solve for the common ratio.

4. The first term of a G.P is 7. Solve the 4th term, if its common ratio is 3

5. Find the product of xy if $x, \frac{3}{2}, \frac{6}{7}, y$ are in the G.P

6. The 2nd and 6th term of a G.P are 81 and 16 respectively, find the value of the 1st term.

7. The first term of a G.P is 22 and the common ratio is 2. Solve for the sum of the first five terms.

8. If the second and fourth terms of a G.P are 8 and 32 respectively. What is the sum of the first four terms?

CHAPTER EIGHT

BINOMIAL THEOREM

AIM: The overall aim of this lecture is to solve problems of Binomial theorem.

OBJECTIVES: At the end of this lecture students should be able to:

1. Construct Pascal's triangle.
2. Define factorial and recognize them.
3. Evaluate factorials by solving them.
4. Define Combination, evaluate and solve combinations.
5. State Binomial theorem and write down binomial expansions for natural number powers.
6. Use the Sigma to solve mathematical problems.

8.1 BINOMIAL THEOREM

8.1.1 Pascal Triangle

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

8.1.2 Binomial Addition.

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

8.1.3 Binomial Subtraction

$(a-b)^0 = 1$

$(a-b)^1 = a-b$

$(a-b)^2 = a^2 + 2a(-b) + (-b)^2 = a^2 - 2ab + b^2$

$(a-b)^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3 =$

$a^3 - 3a^2b + 3ab^2 - b^3$

$(a-b)^4 = a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 +$

$(-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

8.1.4 Factorial

$n! = n \times (n-1) \times (n-2) \ldots\ldots \times (n-k)!$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

8.1.5 Combinations

$$^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

$$^{n}C_{n} = \frac{n!}{(n-n)!n!} = \frac{n!}{0!n!} = 1 \quad \text{and}$$

$$^{n}C_{0} = \frac{n!}{(n-0)!0!} = \frac{n!}{n!0!} = 1$$

$$^{6}C_{2} = \frac{6!}{(6-2)!2!} = \frac{6!}{4!2!} = \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times2!} = \frac{6\times5\times4!}{4!\times2\times1} = 15$$

$$^{-2}C_{2} = \frac{-2!}{(-2-2)!2!} = \frac{-2!}{-4!2!} = \frac{-2\times-3\times-4!}{-4!2\times1} = \frac{6}{2} = 3$$

8.1.6 Formula for Binomial Theorem

If $|a| > |b|$, then

$$(a+b)^n = {}^{n}C_0 a^n b^0 +$$

$$^{n}C_1 a^{n-1} b^1 + {}^{n}C_2 a^{n-2} b^2 + {}^{n}C_3 a^{n-3} b^3 + \dots \dots .. + {}^{n}C_n a^0 b^n$$

$$(a+b)^n = a^n + a^{n-1} b^1 + \frac{n(n-1)}{2!} a^{n-2} b^2 +$$

$$\frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \cdots \dots .. + {}^{n}C_n a^0 b^n$$

$$(a+b)^n = \sum_{r}^{n} {}^{n}C_r a^{n-r} b^r$$

Where n is a positive, negative, integral or any rational.

Binomial expansion

General term of the binomial expansion is given as

$(a+b)^n =$

${}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots \dots .. + {}^nC_n a^{n-n} b^n$

$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} +$

$\frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \cdots + b^n$

Example 1: Expand $(x+y)^7$

Solution:

$(x+y)^7$

$= {}^7C_0 x^7 y^0 + {}^7C_1 x^{7-1} y^1 + {}^7C_2 x^{7-2} y^2 + {}^7C_3 x^{7-3} y^3 +$

${}^7C_4 x^{7-4} y^4 + {}^7C_5 x^{7-5} y^5 + {}^7C_6 x^{7-6} y^6 + {}^7C_7 x^{7-7} y^7 =$

$= x^7 + \frac{7x^6y}{1} + \frac{7\times6x^5y^2}{2!} + \frac{7\times6\times5x^4y^3}{3!} + \frac{7\times6\times5\times4x^3y^4}{4!} +$

$\frac{7\times6\times5\times4\times3x^2y^5}{5!} + \frac{7\times6\times5\times4\times3\times2x^1y^6}{6!} + y^7$

$x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 +$

$21x^2y^5 + 7xy^6 + y^7$

Example 2: Expand $(1+x)^5$

Solution:

$(1+x)^5 =$

${}^5C_0 1^5 x^0 + {}^5C_1 1^{5-1} x^1 + {}^5C_2 1^{5-2} x^2 + {}^5C_3 1^{5-3} x^3 +$

${}^5C_4 1^{5-4} x^4 + {}^5C_5 1^{5-5} x^5$

$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

Exercise: Expand $(2 + 3x)^4$

Example 3: Expand $\left(2 - \frac{x}{2}\right)^3$

Solution:

$\left(2 - \frac{x}{2}\right)^3 =$

${}^3C_0 2^3 \left(-\frac{x}{2}\right)^0 + {}^3C_1 1^{3-1} \left(-\frac{x}{2}\right)^1 + {}^3C_2 1^{3-2} \left(-\frac{x}{2}\right)^2 +$

${}^3C_3 1^{3-3} \left(-\frac{x}{2}\right)^3$

$\left(2 - \frac{x}{2}\right)^3 =$

${}^3C_0 2^3 \left(-\frac{x}{2}\right)^0 + {}^3C_1 1^{3-1} \left(-\frac{x}{2}\right)^1 + {}^3C_2 1^{3-2} \left(-\frac{x}{2}\right)^2 +$

${}^3C_3 1^{3-3} \left(-\frac{x}{2}\right)^3$

$= 8 - 6x + \frac{3}{2}x^2 - \frac{1}{8}x^3$

Example 4: Expand $\left(1 + \frac{1}{3}x\right)^{-2}$

Solution

$\left(1 + \frac{1}{3}x\right)^{-2} =$

${}^{-2}C_0 1^{-2} \left(\frac{x}{3}\right)^0 + {}^{-2}C_1 1^{-2-1} \left(\frac{x}{3}\right)^1 + {}^{-2}C_2 1^{-2-2} \left(\frac{x}{3}\right)^2 +$

$$^{-2}C_3 1^{-2-3}\left(\frac{x}{3}\right)^3 +$$

$$^{-2}C_4 1^{-2-4}\left(\frac{x}{3}\right)^4 + {}^{-2}C_5 1^{-2-7}\left(\frac{x}{3}\right)^5 + \cdots$$

$$\left(1 + \frac{1}{3}x\right)^{-2} = 1 - \frac{2x}{3} + \frac{x^2}{3} - \frac{4x^3}{27} + \frac{5x^4}{81} - \frac{2x^5}{81} + \cdots$$

8.2 The Sigma Notation ($\sum$)

Example 5: Solve for the value of the Sigma Notation $\sum_{r=1}^{6}(3r-1)$

Solution:

$$\sum_{r=1}^{6}(3r-1) = 2 + 5 + 8 + 11 + 14 + 17 = 57$$

8.3 The Sum of the First n Natural Numbers

We look at the sum of the first n non-zero natural numbers

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n$$

$$\sum_{r=1}^{n} r = n + (n-1) + (n-2) + \cdots + 1$$

Add both equations above

$$2\sum_{r=1}^{n} r = (1+n) + (2+n-1) + (3+n-2) + \cdots + (n+1)$$

$$2\sum_{r=1}^{n} r = (1+n) + (1+n) + (1+n) + \cdots + (1+n)$$

$2\sum_{r=1}^{n} r = n(1+n)$

$\sum_{r=1}^{n} r = \frac{n(1+n)}{2}$

Example 6: Solve for the value of $\sum_{r=1}^{n}(8r+5)$

Solution:

$\sum_{r=1}^{n} 8r + \sum_{r=1}^{n} 5 = 8\sum_{r=1}^{n} r + \sum_{r=1}^{n} 5$

$= 8 \times \frac{n(1+n)}{2} + 5n$

$= 4n(1+n) + 5n$

$= 4n + 5n + 4n^2$

$= 4n^2 + 9n$

Assessment:

1. Expand $\left(2 - \frac{x}{3}\right)^8$
2. Expand $\left(1 + \left(\frac{1}{2}x + x^2\right)\right)^4$
3. $\left(1 - \frac{2}{3}x\right)^{-3}$
4. $(2x+3y)^5$

5. Expand using Pascal triangle method $\left(2 + (x + 2x^2)\right)^4$

6. Expand using Formula Method/Binomial expansion Method $\left(1 - \frac{1}{2}x\right)^{-3}$
7. Solve the expression $\sum_{r=1}^{n}(5r + 2)$; $\sum_{r=1}^{n}(4r + 2)$
8. Solve the expression $\sum_{r=1}^{36} r$

CHAPTER NINE

SET

9.1 Algebra of sets:

1. **Associative Properties:**

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

2. **Commutative Properties:**

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

3. **Distributive Properties:**

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. **Identity Property for Union:**

$A \cup \varphi = A$ $A \cap U = A; A \cup U = U; A \cap \emptyset = \emptyset$

5. **Intersection Property of the Empty Set:** $A \cap \varphi = \varphi$

6. **Complement Property:**

$(A^c)^c = A; U^C = \emptyset; A \cup A^c = U; A \cap A^c = \emptyset; \emptyset^C = U$

7. **Idempotent Property:** $A \cup A = A; A \cap A = A$

$A \cup B = n(A) + n(B) - n(A \cap B)$

8. The set Difference:

$A - B = A \cap B^c \ or \ B - A = B \cap A^c$

Example 1: Let A = {2, 4, 6, 8}, B = {1, 2, 3, 4} and C = {3, 6, 9}. What are A ∪ B ∪ C and A ∩ B ∩ C?

Solution: Set A ∪ B ∪ C contains elements which are present in at least one of A, B, and C.

A ∪ B ∪ C = {1, 2, 3, 4, 6, 8, 9}.

Set A ∩ B ∩ C contains no element present in all the sets A, B and C .i.e { } or {∅} .

9.2 Symmetric difference

The symmetric difference of sets A and B is the set of objects that are in one and only one of the sets. The symmetric difference is expressed as A∆B. In curly brace notation:

$A\Delta B = \{(A - B) \cup (B - A)\}$

THE SET DIFFERENCE $A - B = A \cap B^c \ or \ B - A = B \cap A^c$

Example 2: Let A be a set of non-negative multiples of two that are no more than twenty. Let B be the non-negative multiples of three that are no more than twenty.

Solution

$A = \{2,4,6,8,10,12,14,16,18,20\}$

$B = \{3,6,9,12,15,18\}$

$U = \{1,2,3,4,5, \dots \dots \dots \dots \dots .. ,20\}$

$A^c = \{1,3,5,7,9,11,13,15,17,19\}$

$B^c = \{1,2,4,5,7,8,10,11,13,14,16,17,19,20\}$

$A - B = A \cap B^c = \{2,4,8,10,14,16,20\}$

$B - A = B \cap A^c = \{3,9,15\}$

$A \Delta B = \{(A - B) \cup (B - A)\}$

$A \Delta B = \{2, 3, 4, 8, 9, 10, 14, 15, 16, 20\}$

Proposition1: Two sets are equal if and only if each is a subset of the other.

$(A = B) \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$

Proof:

Let the two sets in question be A and B. we assume that A = B. A is a subset of itself, likewise

B i.e. $A \subseteq A$. Since A = B substitute A into the left side of the expression to obtain $B \subseteq A$. Also substitute A on the right side to obtain $A \subseteq B$. Since A = B then A and B are subsets of each other.

Assume that $A \subseteq B$ and $B \subseteq A$, the definition of a subset tells us that any element of A is an element of B. Also, any element of B is an element of A. This means that A and B have equal elements satisfying the definition of set equality. We conclude that A = B

9.3 De Morgan's Law

M and N are given sets. De Morgan's Law states that

(i) $(M \cup N)^c = M^c \cap N^c$, and

(ii) $(M \cap N)^c = M^c \cup N^c$.

Proof:

Let $x \in (M \cup N)^c$; Then x does not belong to M or N . Since x does not belong to M we observe that $x \in M^c$. Also, $x \in N^c$. Since x belongs to both M and N, then $x \in M^c \cap N^c$. We observe that $(M \cup N)^c \subseteq$

$M^c \cap N^c$. Let y $\in M^c \cap N^c$. By the definition of intersection of set, y $\in M^c$ and y $\in N^c$. This implies that y does not belong to M ∪ N, since it is not in any of the set, and so we see that y $\in$ (M ∪ N)^c. This shows that $M^c \cap N^c \subseteq$ (M ∪ N)^c. Apply Proposition 1, we have (M ∩ N)$^c = M^c \cup N^c$

Example 3: U = {x : 0 < x < 20} and B = {x : x ∈ U, x is not divisible by 5}. Find B^c

Solution:

U = {1,2,3,4,…………19} and B = {1,2,3,4,6,7,8,9,11,12,13,14,16,17,18,19}

Then B^c = {5,10,15}

Example 4: U is the set of all integers and the subsets L, M and N is given by

L = {x : x ≤3}, M = {x : -6 < x < 14}, N= {x : -1 ≤ x < 18}; find (i) L∩M (ii) L∩(M∪N^c)

Solution:

L = {…,-2,-1,0,1,2,3}; M = {-5,-4,-3,-2,-1,0,1,2,3,4,5,5,6,7,8,9,10,11,1213} ; N = {-1,0,1,…,17} ; N^c = {…,-4,-3,-2,18,19,20…}

(i) L∩M = {-5,-4,-3,-2,-1,0,…,1,2,3} = {-5≤x≤3}

(ii) M∪N^c = {-5,-4,-3,-2,...,14,19, 20…}

Then, L∩(M∪N^c) = {…,-5,-4….,1,2,3} = { x : x ≤3}

Example 5: Given the universal set U = {1,2,3,4,5,6,7,8,9,10}, M = {2,4,6,8,10} and N = {2,3,6,9}.

Show that $(M \cup N)^c = M^c \cap N^c$ (De-morgan's Law)

Solution:

$M \cup N = \{2,3,4,6,8,9,10\}$, $(M \cup N)^c = \{1,5,7\}$

$M^c = \{1,3,5,7,9\}$, $N^c = \{1,4,5,7,8,10\}$, $M^c \cap N^c = \{1,5,7\}$

Example 6: Given X, Y and Z are subsets of the universal set U = {1,2,3,…,10}, X = {P : 2 ≤ P < 10; P is a prime number}, Y = {n : 1 ≤ n ≤ 8}, Z = {m : 2 ≤ m ≤ 10, m is an even number}.

i. List the elements of X, Y and Z
ii. Draw a Venn diagram to illustrate the given information.
iii. Solve for $X^c \cap Z^c \cap Y$

Solution:

i. X = {2,3,5,7}; Y = {1,2,3,4,5,6,7,8}; Z = {2,4,6,8,10}. In addition $Z^c = \{1,3,5,7,9\}$. $X^c = \{1,2,4,6,8,9,10\}$. $Y^c = \{9,10\}$

ii. $X \cap Y \cap Z = \{2\}$, $X \cap Y \cap Z^c = \{3,5,7\}$, $X \cap Y^c \cap Z = \emptyset \; OR \; \{\;\}$, $X^c \cap Y \cap Z = \{2,4,6,8\}$

iii. $X^c \cap Z^c \cap Y = \{1\}$

Example 7: In a science class of 42 students, each student offers at least one of mathematics and physics. If 22 students offer physics and 28 students offer mathematics, how many students offer subjects, only physics and only mathematics?

Solution:

P - Physics, M – Mathematics. Then $n(P \cup M) = n(P) + n(M) - n(P \cap M)$. $n(P \cup M) = n(U) = 42$, $n(P) = 22$, $n(M) = 28$, $n(P \cap M) = x$

$42 = 22 + 28 - x$

$42 = 50 - x$

$x = 8$ We can also show the proof on the Venn diagram.

Only biology $= 22 - 8$, Only mathematics $= 28 - 8$

9.4 Venn diagram for Set

Example 8: In a group of 120 patients, 72 of them receive treatment at Bayelsa Medical Teaching Hospital, 65 receive treatment at Federal Medical Centre and 53 receive treatment at Government House specialist hospital. If 35 are treated Bayelsa Medical Teaching Hospital and Federal Medical Centre, 30 are treated at Bayelsa Medical Teaching Hospital and Government House specialist hospital, 21 are treated at Federal Medical Centre and Government House specialist hospital with each of the patients are at least treated at one of the Hospitals.

a. Draw a Venn diagram to illustrate this information
b. What is the number of patients treated at (i) all three Hospitals (ii) exactly one of the Hospitals (iii) two or more Hospitals.

Solution:

A) B - Bayelsa Medical Teaching Hospital, F - Federal Medical Centre, G - Government House specialist hospital

$$n(B \cup F \cup G) = n(B) + n(F) + n(G) - n(B \cap F) - n(B \cap G) - n(F \cap G) + n(B \cap F \cap G)$$

$n(U) = n(B \cup F \cup G) = 120, n(B \cap F \cap G) = x, n(B) = 72, n(F) = 65, n(G) = 53, n(B \cap F) = 35, n(B \cap G) = 30, n(F \cap G) = 21,$

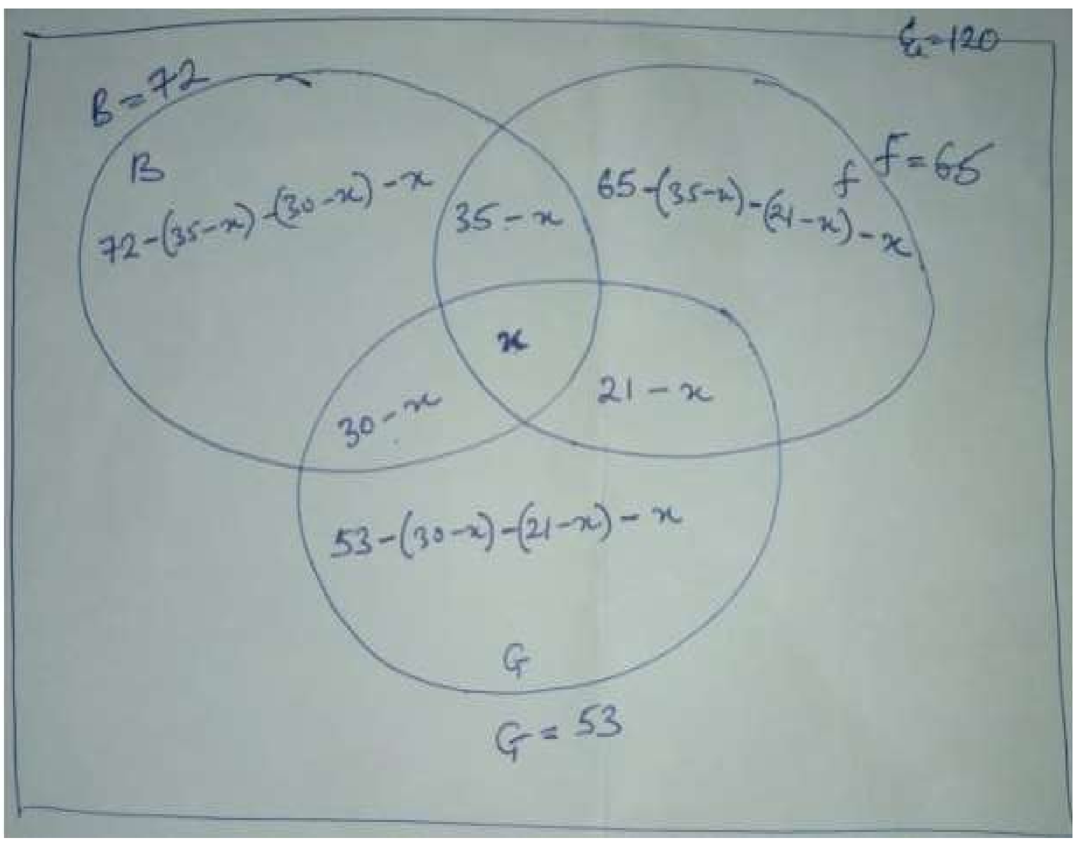
Substituting the parameters into the equation,

$120 = 72 + 65 + 53 - 35 - 30 - 21 + x$

$120 = 104 + x$

$x = 120 - 104 = 16$

Venn Diagram below



B) (ii) Number of patients treated at exactly one of the Hospitals $= n(B)only + n(F)only + n(G)only$

$n(B)only = n(B \cap F^c \cap G^c) = n(B) - n(B \cap F \cap G^c) - n(B \cap G \cap F^c) - n(B \cap F \cap G)$

$= 72 - 19 - 14 - 16 = 23$

$n(F)only = n(F \cap B^c \cap G^c) = n(B) - n(F \cap B \cap G^c) - n(G \cap F \cap B^c) - n(B \cap F \cap G)$

$= 65 - 19 - 5 - 16 = 25$

$n(G)only = n(G \cap B^c \cap F^c) = n(G) - n(G \cap B \cap F^c) - n(G \cap F \cap B^c) - n(B \cap F \cap G)$

$= 53 - 14 - 5 - 16 = 18$

$n(B)only + n(F)only + n(G)only$

$= 23 + 25 + 18 = 66$

(iii) Number of patients treated two or more Hospitals

$= n(B \cap G \cap F^c) + n(G \cap F \cap B^c) + n(B \cap F \cap G^c) + n(B \cap F \cap G) = 14 + 5 + 19 + 16 = 54.$

Assessment:

1. U = {x : 0 < x < 30} and B = {x : x ∈ U, x is not divisible by 4}. Find B^c

2. U is the set of all integers and the subsets L, M and N is given by
   L = {x : x $\leq$5}, M = {x : -5 < x < 18}, N= {x : -2 $\leq$ x < 20}; find (i) L∩M (ii) L∩(M∪N^c)
3. Given the universal set U = {1,2,3,4,5,6,7,8,9,….,20}, M = {2,4,6,8,10,12,14,16,18,20} and N = {2,3,6,9}. Show that $(M \cup N)^c = M^c \cap N^c$ (De-morgan's Law)

CHAPTER TEN

COMPLEX NUMBERS

10.1 Definition of a Complex Number

Complex numbers are derived out of polynomial equations

Given quadratic equation $z^2 - 3z + 2 = 0$, the solution will be expressed as $(z - z_1)(z - z_2) = 0$

Using the formula method $z_{1,2} = \frac{3 \pm \sqrt{(-3)^2 - 4(1 \times 2)}}{2}$

$z_{1,2} = \frac{3 \pm \sqrt{1}}{2} = \frac{3 \pm 1}{2} = (1,2)$

The equation above has a solution no solution.

Given quadratic equation $z^2 + 4z + 5 = 0$, the solution will be expressed as $(z - z_1)(z - z_2) = 0$

Using the formula method $z_{1,2} = \frac{-4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2}$

$z_{1,2} = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm \frac{2i}{2} = -2 \pm i =$

The solution contains both the real and imaginary term known as a complex number.

Complex number is expressed mathematically as $z = x + iy$. With the real term given as x and the imaginary term y.

$i = \sqrt{-1}$.

The real term x expressed as Rez and the imaginary term y is denoted by Imz.

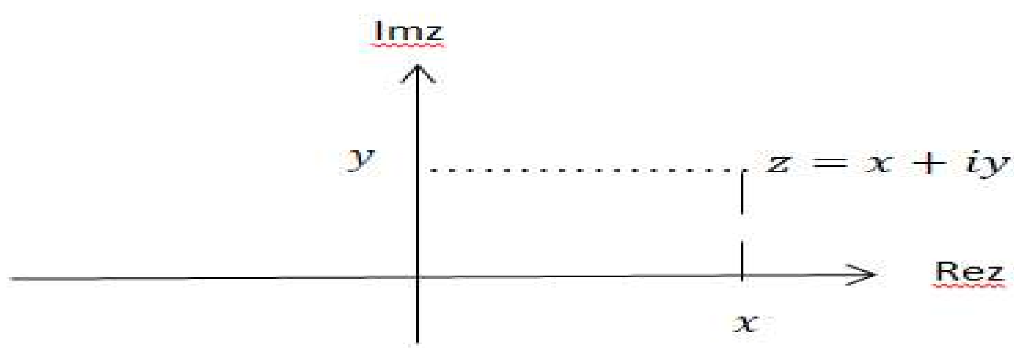
$\sqrt{-4} = \sqrt{-1 \times 4} = 2\sqrt{-1} = 2i$

$z_{1,2} = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm \frac{2i}{2} = -2 \pm i,$

implies that $z_{1,2} = -2 \pm i$ where $x = -2$ and $y = \pm 1$

For compactness, the complex number can be represented in the form $z = (x, y)$.

Argand diagram representing a complex number



The Argand Diagram

10.2 Addition and Subtraction of Complex Number

$z_1 + z_2 =$

$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

$z_1 - z_2 =$

$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

Or in component notation

$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2\,, y_1 + y_2)$

$z_1 + z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2\,, y_1 - y_2)$

The additions of complex numbers two properties

I. associative property

II. commutative property

Associative property:

$z_1 + z_2 = z_2 + z_1 \; for \; all \; z_1\,, z_2 \in z$

Commutative property: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ for all $z_1\,,\; z_2,\; z_3 \in z$

Example 1: Find the sum of the complex number $12 + 3i\,, 1 - 4i\,, -2 + 5i$

Solution:

Sum of the real part: $12 + 1 - 2 = 11$

Sum of the imaginary part: $3i - 4i + 5i = 4i$

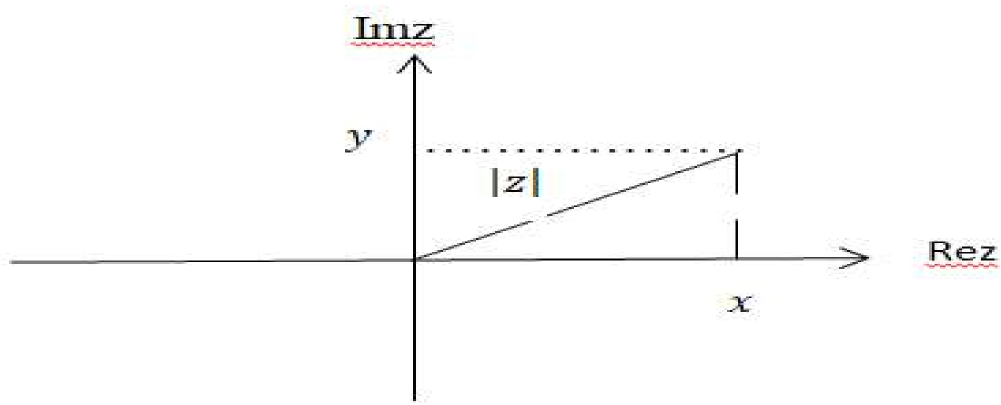
$(12 + 3i) + (1 - 4i) + (-2 + 5i) = 11 + 4i$

10.3 Modulus and Argument

The modulus of a complex number z is expressed $|z| = \sqrt{x^2 + y^2}$ denoting the distance of the corresponding point from the origin in the Argand diagram below.

The argument of a complex number shown on the Argand diagram is expressed as

$\arg z = tan^{-1}\left(\frac{x}{y}\right)$.



The Argand Diagram

When x and y are both negative the $\arg z$ lies in the range $-\pi < \arg z < -\frac{\pi}{2}$ instead of $0 < \arg z < \frac{\pi}{2}$

Example 2: Solve for the modulus and the argument of the complex number $z = 3 - 4i$

Solution:

$|z| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

$\arg z = tan^{-1}\left(-\frac{3}{4}\right)$

Hence z lies in the fourth quadrant $\arg z = -0.75$

10.4 Multiplication of Complex Numbers

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

The multiplication of complex numbers has two properties

I. Commutative

II. Associative

Commutative $z_1 z_2 = z_2 z_1$

Associative $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

The product of two complex numbers has the properties

$|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Example 3: Multiply the complex numbers $z_1 = 1 + 2i$ and $z_2 = -5 - 4i$ and Show that $|z_1 z_2| = |z_1||z_2|$

Solution:

$z_1 z_2 = (1 + 2i)(-5 - 4i)$

$= -5 - 4i - 10i - 8i^2$

$= 3 - 14i$

$|z_1 z_2| = |3 - 14i| = \sqrt{3^2 + (-14)^2} = \sqrt{205}$

$|z_1| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|z_2| = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41}$

$|z_1||z_2| = \sqrt{5} \times \sqrt{41} = \sqrt{205}$

Example 4: Use the geometrical interpretation of multiplication by i, find the product $i(1 - i)$

Solution:

The complex number $1 - i$ has the argument $-\frac{\pi}{4}$ and modulus $\sqrt{2}$.

$i(1 - i) = 1 + i$ has the argument $+\frac{\pi}{4}$ and modulus $\sqrt{2}$

Hence the complex number with modulus of $\sqrt{2}$ and argument $+\frac{\pi}{4}$ is represented as $1 + i$

10.5 Complex Conjugate

If $z = x + iy$, hence its conjugate z^* is expressed as

$z^* = x - iy$

There will be no imaginary part of the complex number when the complex number z is multiplied by its conjugate

$$(x+iy)(x-iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 = |z|^2$$

Addition or Subtraction of conjugate complex number

$$(z_1 \pm z_2)^* = {z_1}^* \pm {z_2}^*$$

Multiplication of conjugate complex number

$$(z_1 z_2)^* = {z_1}^* \, {z_2}^*$$

Division of conjugate complex number

$$\left(\frac{z_1}{z_2}\right)^* = \frac{{z_1}^*}{{z_2}^*}$$

$$(z^*)^* = z$$

$$z + z^* = 2x = 2\, Re\, z$$

$$z - z^* = 2iy = 2i\, Im\, z$$

$$\frac{z}{z^*} = \left(\frac{x^2-y^2}{x^2+y^2}\right) + i\left(\frac{2xy}{x^2+y^2}\right)$$

Exercise: What is the complex conjugate of $z = a + i + 5ib$ Ans. $a - i(1+5b)$

10.6 Division of Complex Numbers.

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} =$$

$$\frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ While $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Example 6: Solve for the quotient $z = \frac{4-3i}{-2+5i}$

Solution:

$$z = \frac{(4-3i)(-2-5i)}{(-2+5i)(-2-5i)} = \frac{-23-21i}{17} = -\frac{23}{17} - \frac{21}{17}i$$

10.7 Polar Representation of Complex Numbers

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{z_1}e^{z_2} = e^{z_1+z_2}$$

Given $z = i\theta$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\,\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\,\frac{\theta^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) - \cdots + i\theta - i\,\frac{\theta^3}{3!} + i\,\frac{\theta^5}{5!} - \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

Hence, $e^{i\theta} = \cos\theta + i\sin\theta$

The equation above is referred to as Euler's Equation

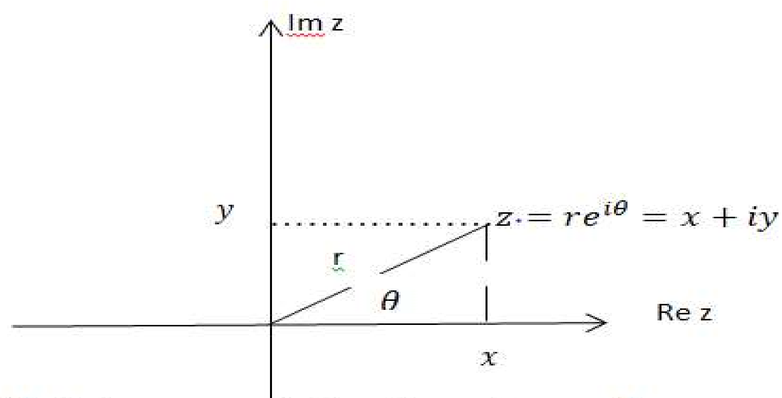
Also, $e^{in\theta} = \cos n\theta + i \sin n\theta$

$re^{i\theta} = \text{r}\cos\theta + i\,\text{r}\sin\theta = x + iy$

A complex number in polar form is expressed as $z = re^{i\theta} = x + iy$

We Identify r with $|z|$ and θ with $\arg z$. This simplifies the representation of the modulus and argument as the main reason using polar representation. θ ranges from $-\pi < \theta \leq \pi$ with θ rotating by $2n\pi + \theta$

$re^{i\theta} = re^{i(\theta+2n\pi)}$



The Polar representation of complex numbers

10.8 Multiplication and Division in Polar Form.

Multiplication.

$z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

$|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$

Division.

$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

$|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

10.9 De Moivre's Theorem

$\left(e^{i\theta}\right)^n = e^{in\theta}$ this implies that

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z = re^{i\theta}$

$z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$

$z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$z^n - \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$

When n = 1

$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Example 7: Express $\sin 4\theta$ and $\cos 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$

Solution:

Using de Moivre's theorem

$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

Using Binomial expansion

$(\cos\theta + i\sin\theta)^4 = cos^4\theta + 4icos^3\theta\sin\theta + 6i^2cos^2\theta sin^2\theta + 4i^3\cos\theta\, sin^2\theta + i^4sin^4\theta$

$(\cos\theta + i\sin\theta)^4 = cos^4\theta + 4icos^3\theta\sin\theta - 6cos^2\theta sin^2\theta - 4i\cos\theta\, sin^2\theta + sin^4\theta$

$(\cos\theta + i\sin\theta)^4 = cos^4\theta + sin^4\theta - 6cos^2\theta sin^2\theta + i(4cos^3\theta\sin\theta - 4\cos\theta\, sin^2\theta)$

$\cos 4\theta = cos^4\theta + sin^4\theta - 6cos^2\theta sin^2\theta = cos^4\theta + sin^4\theta - 6cos^2\theta(1 - cos^2\theta)$

$\cos 4\theta = 7cos^4\theta + sin^4\theta - 6cos^2\theta$

$\sin 4\theta = 4cos^3\theta\sin\theta - 4\cos\theta\sin\theta$

$\sin 4\theta = 4\cos\theta\sin\theta\,(cos^2\theta - 1)$

$\sin 4\theta = 4\cos\theta\sin\theta\, sin^2\theta = 4\cos\theta\, sin^3\theta$

Example 8: Solve the expression for $cos^3\theta$ in terms of $\cos 3\theta$ and $\cos\theta$

Solution:

$$cos^3\theta = \frac{1}{2^3}\left(z + \frac{1}{z}\right)^3 = \frac{1}{8}\left(z^3 + 3z + \frac{3}{z} + \frac{3}{z^3}\right) =$$

$$\frac{1}{8}\left(z^3 + \frac{1}{z^3}\right) + \frac{3}{8}\left(z + \frac{1}{z}\right)$$

$$cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

10.10 nth root of Unity

The equation $z^2 = 1$ has the solution $z = \pm 1$.

$z^n = 1$. Where the equation has n solutions.

$z^n = e^{2ik\pi}$ Where k is any integer. Taking the nth root of both sides we have

$z = e^{2ik\pi/n}$

Example 9: Solve the solution to the equation $z^3 = 1$

Solution:

$z = e^{2ik\pi/3}$

$k = 0;\ z_1 = e^{0i} = 1$

$k = 1;\ z_2 = e^{2i\pi/3}$

$k = 2;\ z_3 = e^{4i\pi/3}$

$k = 3;\ z_4 = e^{6i\pi/3} = 1 = z_1$

Assessment

1. Solve the expression for $cos^4\theta$ in terms of $\cos 4\theta$ and $\cos\theta$
2. Express $\sin 3\theta$ and $\cos 3\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$
3. Solve for the quotient $z = \frac{2-3i}{-5+7i}$
4. Multiply the complex numbers $z_1 = 3 + 5i$ and $z_2 = -1 - 3i$ and Show that $|z_1 z_2| = |z_1||z_2|$
5. Solve for the modulus and the argument of the complex number $z = 5 - 7i$
6. Find the sum of the complex number $10 + 7i$, $5 - 4i$, $-3 + 4i$

CHAPTER ELEVEN

Mathematical Induction

11.1 Mathematical Induction Principle

Given the statement $P(n)$ involving natural number n such that

(i) *The statement is true for* $n = 1$, i.e., $P(1)$ *is true, and*

(ii) *If the statement is true for* $n = k$, *where* k *is some positive integer, then the statement is also true for* $n = k + 1$, i.e., *truth of* $P(k)$ *implies the truth of* $P(k + 1)$.

Then, $P(n)$ *is true for all natural numbers* n.

Mathematical induction principle shows that a given proposition such as inequality, equality, formula,… is true for every positive integer number greater than or equal to some integer N.

P (n) denotes the preposition where n is a positive integer.

The proof is done in two parts:

Part 1: The proposition P (n) must be true for the smallest possible value of the positive integer n.

Part 2: P (k) is assumed to be true as well as P (k+1).

Example 1

Show that

$$\mathbf{1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}}$$

for all positive integers n by using the principle of mathematical induction.

Solution:

we define the statement P(n) as

$$1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$$

Part 1: Show that p (1) is true on the left part and right part of the equation.

Left part = 1

Right part $= \frac{1\,(1+1)}{2} = 1$

The two part of the statement are equivalent, therefore P(1) is true.

Part 2: Assume P(k) to be true

$$1 + 2 + 3 + \ldots + k = \frac{k\,(k+1)}{2}$$

we then show that P(k + 1) is true. This is done by the addition of k + 1 to both parts of the statement P(k)

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{k\,(k+1)}{2} + (k+1)$$

$$= \frac{k\,(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

Therefore, the final statement is written as

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+2)(k+1)}{2}$$

Which is the statement P(k + 1).

Example 2

Show that

$$\mathbf{1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n\,(n+1)(2n+1)}{6}}$$

For all positive integers n by using the principle of mathematical induction

<u>Solution:</u>

Statement P (n) is defined by

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n\,(n+1)(2n+1)}{6}$$

Part 1: Show that p (1) is true on the left part and right part of the equation.

Left Part= $1^2 = 1$

Right Part $= \frac{\mathbf{1\,(1+1)[2(1)+1]}}{\mathbf{6}} = \frac{\mathbf{1(2)(3)}}{\mathbf{6}} = \frac{\mathbf{6}}{\mathbf{6}} = \mathbf{1}$

The two part of the statement are equivalent, therefore P(1) is true.

Part 2: Assume P(k) to be true

$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{\boldsymbol{k\,(k+1)(2k+1)}}{\mathbf{6}}$

we then show that P(k + 1) is true. This is done by the addition of $(k + 1)^2$ to both parts of the statement P(k)

$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = = \frac{\boldsymbol{k\,(k+1)(2k+1)}}{\mathbf{6}} + (k + 1)^2$

Set common denominator and factor k + 1 on the right side

$= \frac{\boldsymbol{(k+1)[k\,(2k+1)+6(k+1)]}}{\mathbf{6}}$

$= \frac{\boldsymbol{(k+1)[2k^2+k+6k+6]}}{\mathbf{6}}$

$= \frac{\boldsymbol{(k+1)[2k^2+7k+6]}}{\mathbf{6}}$

Factorize $\boldsymbol{2k^2 + 7k + 6}$.

$$= \frac{(\boldsymbol{k+1})[(\boldsymbol{k}+\mathbf{2})(\mathbf{2}\boldsymbol{k}+\mathbf{3})]}{\mathbf{6}}$$

From statement P(k), we have shown that

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(\boldsymbol{k+1})[(\boldsymbol{k}+\mathbf{2})(\mathbf{2}\boldsymbol{k}+\mathbf{3})]}{\mathbf{6}}$$

Which is the statement P(k + 1).

Example 3

Prove that

$$\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3} = \frac{n^2(n+1)^2}{4}$$

for all positive integers n using the principle of mathematical induction

<u>Solution</u>

The Statement P (n) is defined as

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

PART 1: Let us prove that P (1) is true.

Left Part = $1^3 = 1$

Right Part = $\frac{1^2(1+1)^2}{4} = 1$

P (1) is true.

PART 2: Assume P (k) is true

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$$

add $(k+1)^3$ to both sides of the equation

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} +$

$(k+1)^3$

factor out $(k+1)^2$

$= (k+1)^2 \left[\frac{k^2}{4} + (k+1)\right]$

$= (k+1)^2 \left[\frac{k^2+4(k+1)}{4}\right]$

$= (k+1)^2 \left[\frac{k^2+4k+1}{4}\right]$

$= (k+1)^2 \left[\frac{(k+2)^2}{4}\right]$

Given the statement P (k) we have shown that

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = (k+1)^2 \left[\frac{(k+2)^2}{4}\right]$

Which is the statement P(k + 1).

Example 4

Prove De Moivre's theorem using the theorem of mathematical induction

$[\boldsymbol{R}\ (\boldsymbol{cos\ x}\ +\ \boldsymbol{i\ sin\ x})]^{\boldsymbol{n}} = \boldsymbol{R^n}(\mathbf{cos\ nt}\ +\ \mathbf{i\ sin\ nt})$

for n a positive integer.

Solution:

PART 1: For $n = 1$

$[R (\cos x + i \sin x)]^{1} = R^{1}(\cos 1*x + i \sin 1*x)$

It can easily be seen that the two sides are equal.

PART 2: Assuming the theorem is true for $n = k$, Then

$[R (\cos x + i \sin x)]^{k} = R^{k}(\cos kx + i \sin kx)$

Multiply both sides of the equation with $R (\cos x + i \sin x)$

$[R (\cos x + i \sin x)]^{k} R (\cos x + i \sin x) = R^{k}(\cos kx + i \sin kx) R (\cos x + i \sin x)$

$[R (\cos x + i \sin x)]^{k+1} = R^{k+1} [(\cos kx \cos x - \sin kx \sin x) + i (\sin kx \cos x + \cos kx \sin x)]$

Applying trigonometric identities the expressions $(\cos kx \cos x - \sin kx \sin x)$ and $(\sin kx \cos x + \cos kx \sin x)$ is re written as

Trigonometric addition of Cosine

$\cos (kx + x) = \cos (k + 1) x = (\cos kx \cos x - \sin kx \sin x)$

Trigonometric addition of Sine

$(\sin kx \cos x + \cos kx \sin x) = \sin (kx + x) = \sin (k + 1)x$

Substitute the above into the last equation to obtain

$[R (\cos x + i \sin x)]^{k+1} = R^{k+1} [\cos (k + 1)x + \sin(k + 1)x]$

the statement P(n) is true for n = 1 and if assumed true for n = k, then it is true for n = k + 1.

Assessment

1. Prove that for any positive integer number n; $\mathbf{n^3 + 2n}$ is divisible by 3
2. Prove that $\mathbf{3^n > n^2}$ for n = 1, n = 2 prove that $3^n > n^2$ for n a positive integer greater than 2 using the principle of mathematical induction.
3. Prove that $\mathbf{n! > 2^n}$ for n a positive integer greater than or equal to 4. (Note: n! is n factorial and is given by 1 * 2 * ...* (n-1)*n.)
4. Prove that $\mathbf{3^n > n^2}$ for n = 1, n = 2 and use the mathematical induction to prove that $3^n > n^2$ for n a positive integer greater than 2.

CHAPTER TWELVE
TRIGONOMETRY

12.1 Definition of Trigonometry

Trigonometry is the branch of mathematics that deals with the measurement of the sides and angle of a plane triangle. A major part of trigonometry is trigonometric functions. Trigonometry has applications in mensuration, mechanics physics, surveying, medicine, engineering, sciences, astronomy etc.

12.2 Types of Trigonometry:

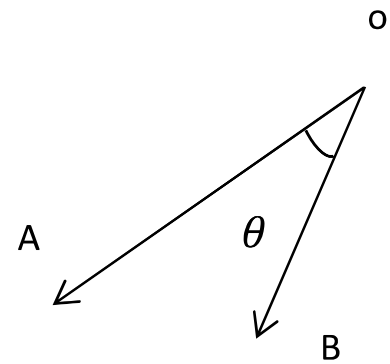
The types of trigonometry are

(1) Plane Trigonometry

(2) Spherical Trigonometry

1. Plane Trigonometry: This aspect deals with angles, triangles and other figures lying in a plane.
2. Spherical Trigonometry: This aspect deals with triangles that lies on a sphere and the sides of which are circular arcs.

12.3 Angle:



An angle could be defined when two line meet at a point described as θ (<AOB) which could be either positive or negative

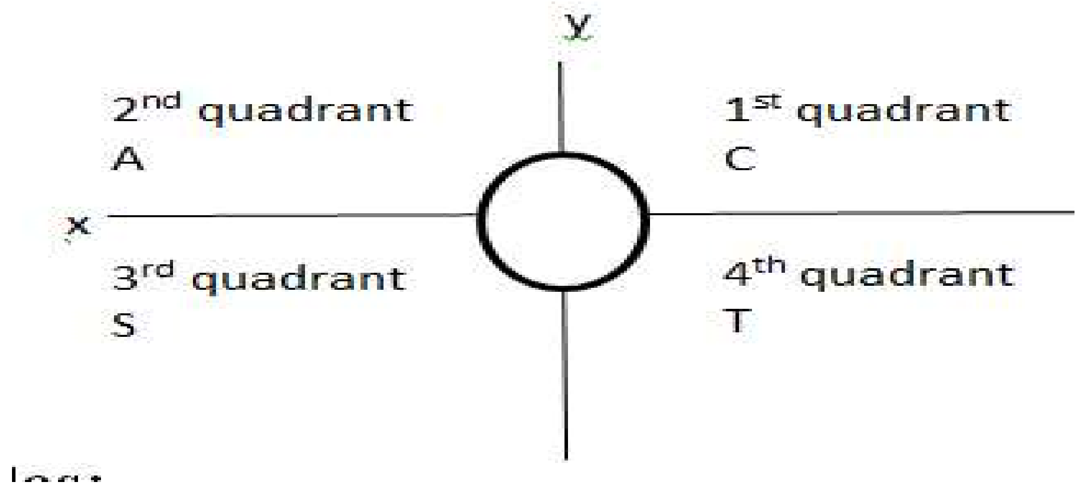
Quadrants:

The 1st quadrant is from 0^o to 90^o in an anti-clockwise direction and from -270^o to -360^o a clockwise direction.

The 2nd quadrant is from 90^o to 180^o in an anti-clockwise direction and from -180^o to -270^o in a clockwise direction.

The 3rd quadrant is from 180^o to 270^o in an anti-clockwise direction and from -90^o to -180^o in a clockwise direction.

The 4th quadrant is from 270^o to 360^o in an anti-clockwise direction and from – 0° to – 90° in clockwise direction.



12.4 Measurement of Angles:

Angles are measured in degree or radian

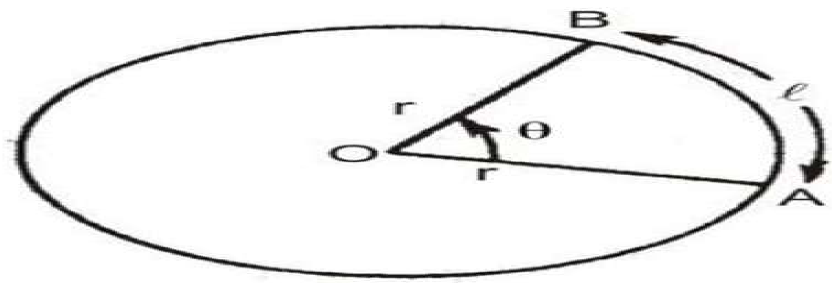
1 right angle is the equivalent of $\frac{360^0}{4} = 90^0$

1 minute is the equivalent of 60 seconds

1 degree is the equivalent of 60 minutes (3600 seconds = 1 hour)

1 circle is the equivalent of 360^0 = 21,600 minutes = 1,296,000 sec

Circular measure system (Radian):



The radian is the angle that subtends at the Centre of the circle by an arc equivalent in length to the radius of the circle. The arc AB is equal to the radius OB of the circle.

12.5 Measurement of Degree and Radian:

The circumference of a circle is $2\pi r$ where r is the radius

2π radians = 360^o

π Radians = 180^o

1 radian = $\frac{180}{\pi}$

$1^0 = \frac{\pi}{180}$

A complete revolution is 360^0.

Examples: Convert 140^0 into radians

Solution.

$140^0 = 140 \times \frac{\pi}{180} = 7 \times \frac{\pi}{9} = 2 \times \frac{3.142}{3} =$

$2.444\, radians$

Example: Convert $40^o\ 30'\ 25''$ into Radian measure.

Solution

$$40^o\ 30'\ 25'' = 40^0 + \frac{30}{60} + \frac{25}{3600} = 40^0 + \frac{382^0}{900} = 40^0 + \frac{1800^0 + 25^0}{3600} = 40^0 + \frac{1825^0}{3600} = \frac{145825^0}{3600} \times \frac{\pi}{180} =$$

$0.7071\ radians$

Example: Convert 20.589^0 to degree, minutes and seconds.

Solution:

$$20.589^0 = 20^0 + (0.589 \times 60)'$$

$$= 20^0 + 35' + (0.34 \times 60)''$$

$$= 20^0 + 35' + 21.6''$$

$$= 20^0 35' 22''$$

Example: Express in degree $\frac{5}{3}\pi$

Solution:

$$\frac{5}{3}\pi = \frac{5}{3}\pi \times \frac{180}{\pi} = 300^0$$

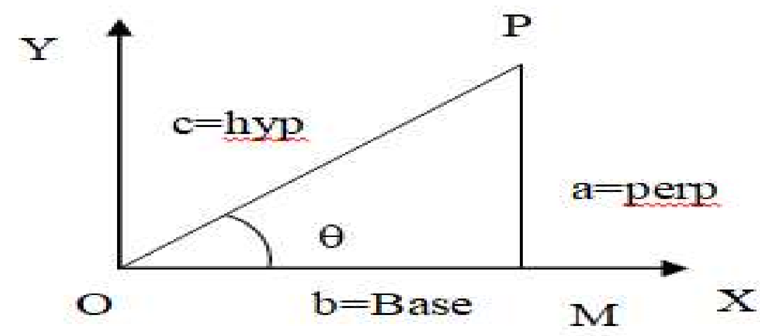
Exercise:

1. Convert the following to Radian measure

(i) 210^o (ii) 540^o (iii) $42^o\ 36'\ 12''$ (iv) $24^o 32' 30''$

2. Convert the following to degree measure

(i) $\frac{5}{4}\pi$ (ii) $\frac{2}{3}\pi$ (iii) 5.52 radian (iv) 1.30 radian



12.6 PYTHAGORAS THEOREM

$c^2 = a^2 + b^2$

$a^2 = c^2 - b^2$

$b^2 = c^2 - a^2$

12.7 Trigonometric Function and Ratios:

Line OX revolves to produce angle θ. Take point P on the final line and draw a perpendicular line PM from P on line OX to form <XOP = θ, where θ may be is measured in degree or radians.

Now OMP is a right angled triangle, we can form the six ratios as follows therefore these ratios called

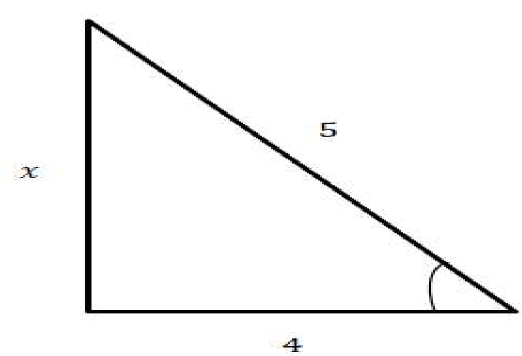
Trigonometric ratios or trigonometric functions of angle $\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}$

$c = hypothenus, a = opposite, b = adjacent$

$\sin\theta = \frac{Opposite}{Hypothenus} = \frac{PM}{PO} = \frac{a}{c}; \cos\theta = \frac{Adjacent}{Hypothenus} = \frac{MO}{PO} = \frac{b}{c}; \tan\theta = \frac{Opposite}{Adjacent} = \frac{a}{b}.$

$cosecant\ \theta = \frac{1}{\sin\theta} = \frac{Hypothenus}{Opposite} = \frac{c}{a}$; $\text{secant}\,\theta = \frac{1}{\cos\theta} = \frac{Hypothenus}{Adjacent} = \frac{c}{b}$; $\text{cotangent}\,\theta = \frac{1}{\tan\theta} = \frac{Adjacent}{Opposite} = \frac{b}{a}$

Example: Use the diagram below to solve for x, $\sin\theta$; $\cos\theta$; $\tan\theta$; $\sec\theta$; $cosec\ \theta$; $\cot\theta$

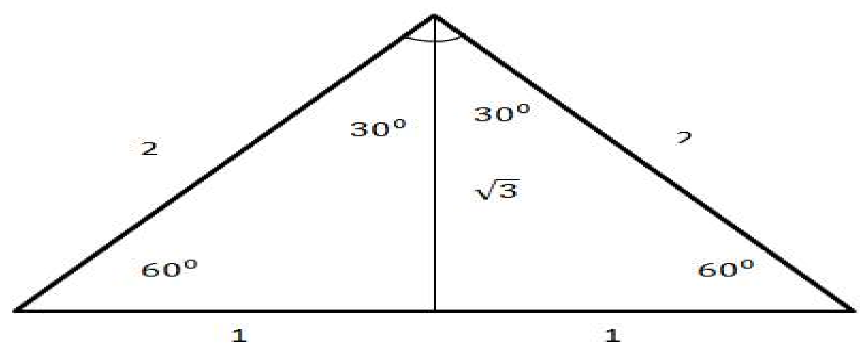


Solution

From Pythagoras theorem, $x^2 = 5^2 - 4^2 = 25 - 16 = 9; x = 3$

$\sin\theta = \frac{3}{5}; \cos\theta = \frac{4}{5}; \tan\theta = \frac{3}{4}; cosec\ \theta = \frac{5}{3}; \sec\theta = \frac{5}{4}; \cot\theta = \frac{4}{3}$

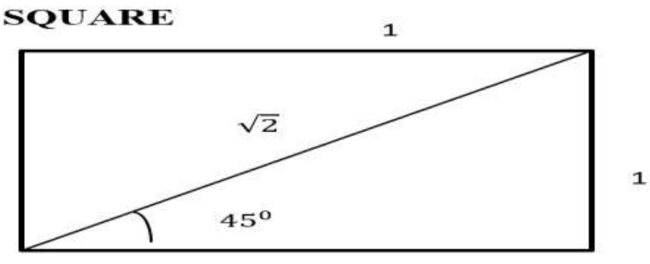
EQUILATERAL TRIANGLE



$\sin 60^0 = \frac{\sqrt{3}}{2}; \cos 60^0 = \frac{1}{2}; \tan 60^0 = \sqrt{3}; cosec\ 60^0 = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}; \sec 60^0 = 2; \cot 60^0 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 30^0 = \frac{1}{2}; \cos 30^0 = \frac{\sqrt{3}}{2}; \tan 30^0 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}; cosec\ 30^0 = 2; \sec 30^0 = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}; \cot 30^0 = \sqrt{3}$

SQUARE



$\sin 45^0 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$; $\cos 45^0 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$; $\tan 45^0 = 1$; $cosec\ 45^0 = \sqrt{2}$; $\sec 45^0 = \sqrt{2}$; $\cot 45^0 = 1$

ANGLE	$\mathbf{0^0}$	$\mathbf{30^0}$	$\mathbf{45^0}$	$\mathbf{60^0}$	$\mathbf{90^0}$
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tangent	0	1	1	$\sqrt{3}$	∞
Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
Cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

Example. Given that $\theta = 30^0$, verify that

a. $\sin 3\theta = 3\sin\theta - 4\, sin^3\,\theta$

b. $\cos 3\theta = 4\, cos^3\theta - 3\cos\theta$

Solution:

a. $\sin 3\theta = \sin 3 \times 30^0 = \sin 90^0 = 1$

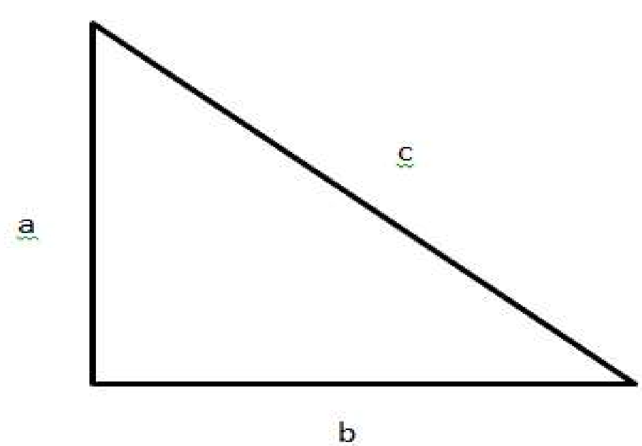
$3\sin\theta - 4\,sin^3\,\theta = \frac{3}{2} - \frac{4}{8} = \frac{8}{8} = 1$

b. $\cos 3\theta = \cos 90 = 0$

$3cos^3\theta - 3\cos\theta = 4 \times \frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} =$

0

12.8 TRIGONOMETRIC IDENTITY



$c^2 = a^2 + b^2$

$\frac{c^2}{c^2} = \frac{a^2}{c^2} + \frac{b^2}{c^2}$

$1 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$

$1 = sin^2\theta + cos^2\theta$ ………..*

$\frac{1}{cos^2\theta} = \frac{sin^2\theta}{cos^2\theta} + \frac{cos^2\theta}{cos^2\theta}$

$sec^2\theta = 1 + tan^2\theta$ ....**

$$\frac{1}{sin^2\theta} = \frac{sin^2\theta}{sin^2\theta} + \frac{cos^2\theta}{sin^2\theta}$$

$Cosec^2\theta = 1 + cot^2\theta$ ....***

Example: show that $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2cosec^2\theta$

Solution:

$$\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}$$

$$= \frac{1+\cos\theta+1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1}{1-cos^2\theta}$$

$$= \frac{1}{sin^2\theta}$$

Example: Show that $\sin\theta\,(1+\tan\theta) + \cos\theta\,(1+\cot\theta) = \sec\theta + cosec\,\theta$

Solution.

LHS = $\sin\theta\,(1+\tan\theta) + \cos\theta\,(1+\cot\theta)$

$$= \sin\theta\left(1+\frac{\sin\theta}{\cos\theta}\right) + \cos\theta\left(1+\frac{\cos\theta}{\sin\theta}\right)$$

$$= \sin\theta + \frac{sin^2\theta}{\cos\theta} + \cos\theta + \frac{cos^2\theta}{\sin\theta}$$

$$= \frac{sin^2\theta\cos\theta + sin^3\theta + cos^2\theta\sin\theta + cos^3\theta}{\sin\theta\cos\theta}$$

$$= \frac{sin^2\theta(\sin\theta\cos\theta) + cos^2\theta(\sin\theta+\cos\theta)}{\sin\theta\cos\theta}$$

$= \frac{(sin^2\theta+cos^2\theta)(\sin\theta+\cos\theta)}{\sin\theta\cos\theta}$

$= \frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta} = \frac{\sin\theta}{\sin\theta\cos\theta} + \frac{\cos\theta}{\sin\theta\cos\theta} = \frac{1}{\cos\theta} + \frac{1}{\sin\theta} =$ $\sec\theta\, cosec\ \theta$

Example: Show that $tan^2\theta - sin^2\theta = sin^4\theta sec^2\theta$

Solution:

$tan^2\theta - sin^2\theta = \frac{sin^2\theta}{cos^2\theta} - sin^2\theta = sin^2\theta sec^2\theta - sin^2\theta = sin^2\theta(sec^2\theta - 1) = sin^2\theta tan^2\theta = sin^2\theta\left(\frac{sin^2\theta}{cos^2\theta}\right) = sin^4\theta sec^2\theta$

Example: Given that $u = \frac{1+\sin\theta}{\cos\theta}$ show that $\frac{1}{u} = \frac{1-\sin\theta}{\cos\theta}$

Solution:

$\frac{1}{u} = \frac{\cos\theta}{1+\sin\theta} = \frac{\cos\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta} = \frac{\cos\theta(1-\sin\theta)}{1-sin^2\theta} = \frac{\cos\theta(1-\sin\theta)}{cos^2\theta} = \frac{1-\sin\theta}{\cos\theta}$

Example: Solve the expression (in terms of p and q) of $\frac{P\cos\theta+q\ sin\ \theta}{pcos\ \theta-q\sin\theta}$ where $\cot\theta = \frac{p}{q}$

Solution.

$$\frac{p\cos\theta+q\sin\theta}{p\cos\theta-q\sin\theta}=\frac{p\frac{\cos\theta}{\sin\theta}+q}{p\frac{\cos\theta}{\sin\theta}-q}=\frac{p\cot\theta+q}{p\cot\theta-q}=\frac{\frac{p^2}{q}+q}{\frac{p^2}{q}-q}=\frac{p^2+q^2}{p^2-q^2}$$

Example. Given $\tan\theta=\frac{4}{5}$, solve for the value of $\frac{2\sin\theta+3\cos\theta}{4\cos\theta+3\sin\theta}$,

Solution.

$$\frac{2\sin\theta+3\cos\theta}{4\cos\theta+3\sin\theta}=\frac{2\frac{\sin\theta}{\cos\theta}+3}{4+3\frac{\sin\theta}{\cos\theta}}=\frac{2\tan\theta+3}{4+3\tan\theta}=\frac{\frac{8}{5}+3}{4+\frac{12}{5}}=\frac{\frac{23}{5}}{\frac{32}{5}}=\frac{23}{32}$$

TRIGONOMETRIC EQUATION:

Example: Solve the equation $8\cos^2\theta+10\sin\theta-11=0$ for $0^0\le\theta\le360^0$.

Solution:

$$\sin^2\theta+\cos^2\theta=1$$

$$\cos^2\theta=1-\sin^2\theta$$

$$8(1-\sin^2\theta)+10\sin\theta-11=0$$

$$8-8\sin^2\theta+10\sin\theta-11=0$$

$$8\sin^2\theta-10\sin\theta+3=0$$

$$8\sin^2\theta-6\sin\theta-4\sin\theta+3=0$$

$$2\sin\theta\,(4\sin\theta-3)-1(4\sin\theta-3)=0$$

$(4\sin\theta - 3)(2\sin\theta - 1) = 0$

$4\sin\theta - 3 = 0$

$\sin\theta = \frac{3}{4} \Rightarrow \theta = \sin^{-1}\frac{3}{4} = 48.6^0 \cong 49^0$

$2\sin\theta - 1 = 0 \Rightarrow \theta = \sin^{-1}\frac{1}{2} = 30^0$

Example: show that $\cos\theta$ is not imaginary given that

$2sin^2\theta - 3\cos\theta - 6 = 0$

Solution.

$3sin^2\theta - 2\cos\theta - 5 = 0$

$3(1 - cos^2\theta) - 2\cos\theta - 5 = 0$

$-3cos^2\theta - 2\cos\theta - 2 = 0$

$3cos^2\theta + 2\cos\theta + 2 = 0$

$\cos\theta = \frac{-2\pm\sqrt{-20}}{4} = \frac{-2\pm\sqrt{-20}}{4}$ Hence $\cos\theta$ is imaginary.

12.9 Negative Angle Identities

Sin $(-\theta) =$ $-\sin\theta$	**Sin $(90-\theta)$ = cos θ**	**Sin $(90+\theta)$ = cos θ**	**Sin $(180-\theta)$ = sin θ**	**Sin $(180+\theta)$ = -sin θ**	**Sin $(360-\theta)$ = -sin θ**

cos $(-\theta) = \cos\theta$	**cos $(90-\theta) = \sin\theta$**	**cos $(90+\theta) = -\sin\theta$**	**cos $(180-\theta) = -\cos\theta$**	**cos $(180+\theta) = -\cos\theta$**	**cos $(360-\theta) = \cos\theta$**
tan $(-\theta) = -\tan\theta$	**tan $(90-\theta) = \cot\theta$**	**tan $(90+\theta) = -\cot\theta$**	**tan $(180-\theta) = -\tan\theta$**	**tan $(180+\theta) = \tan\theta$**	**tan $(360-\theta) = -\tan\theta$**
cosec $(-\theta) = -\text{cosec}\,\theta$	**cosec $(90-\theta) = \sec\theta$**	**cosec $(90+\theta) = \sec\theta$**	**cosec $(180-\theta) = \text{cosec}\,\theta$**	**cosec $(180+\theta) = -\text{cosec}\,\theta$**	**cosec $(360-\theta) = -\text{cosec}\,\theta$**
sec $(-\theta) = \sec\theta$	**sec $(90-\theta) = \text{cosec}\,\theta$**	**sec $(90+\theta) = -\text{cosec}\,\theta$**	**sec $(180-\theta) = -\sec\theta$**	**sec $(180+\theta) = -\sec\theta$**	**sec $(360-\theta) = \sec\theta$**
cot $(-\theta) = -\cot\theta$	**cot $(90-\theta) = \tan\theta$**	**cot $(90+\theta) = -\tan\theta$**	**cot $(180-\theta) = -\cot\theta$**	**cot $(180+\theta) = -\text{cosec}\,\theta$**	**cot $(360-\theta) = -\cot\theta$**

12.10 Sum and differences of SINES, COSINES AND TANGENTS

$\sin(A+B) = \sin A \cos B + \cos A \sin B$;

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$;

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$;

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

12.11 Trigonometrical ratios of Multiple and sub Multiple Angles

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2cos^2 A - 1 = 1 - 2sin^2 A = cos^2 A - sin^2 A$

$\tan 2A = \frac{2 \tan A}{1 - tan^2 A}$

Furthermore,

$\sin 3A = 3 \sin A - 4sin^3 A$

$\cos 3A = 4cos^3 A - 3 \cos A$

$\tan 3A = \frac{3 \tan A - tan^3 A}{1 - 3tan^2 A}$

Furthermore

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

Examples: Solve the angles in surd form.

(i) $\cos 75^0$ (ii) $\cos 240^0$ (iii) $\sin 105^0$ (iv) $\sin 195^0$

(v) $\cos 15^0$ (vi) $\sin 15^0$ (vii) $\tan 105^0$ (viii) $\tan 15^0$

Solution:

(i) $\cos 75^0 = \cos(45^0 + 30^0) =$

$\cos 45^0 \cos 30^0 - \sin 45^0 \sin 30^0 =$

$\frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}.\frac{1}{2} = \frac{\sqrt{2\times3}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} =$

$\frac{1}{4}(\sqrt{6} - \sqrt{2})$

(ii) $\cos 240^0 = \cos(180^0 + 60^0) =$

$\cos 180^0 \cos 60^0 - \sin 180^0 \sin 60^0 =$

$$\left(-1 \times \frac{1}{2}\right) - \left(0 \times \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}$$

(iii) $\sin 105^0 = \sin(45^0 + 60^0) =$

$\sin 45^0 \cos 60^0 + \cos 45^0 \sin 60^0 = \frac{\sqrt{2}}{2} . \frac{1}{2} +$

$\frac{\sqrt{2}}{2} . \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{1}{4}(\sqrt{2} + \sqrt{6})$

(iv) $\tan 15^0 = \tan(60^0 - 45^0) =$

$\frac{\tan 60^0 - \tan 45^0}{1 + \tan 60^0 \tan 45^0} = \frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{\sqrt{3}-1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} =$

$\frac{\sqrt{3}-3-1+\sqrt{3}}{1-3} = \frac{2\sqrt{3}-4}{-2} = \frac{\sqrt{3}-2}{-1} = 2 - \sqrt{3}$

12.12 Formula for Multiple Angles

Proof should be shown

$\sin 2A = 2 \sin A \cos A$; $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$

$\cos 2A = 2cos^2 A - 1$; $\cos A = 2cos^2 \frac{A}{2} - 1$

$\tan 2A = \frac{2 \tan A}{1 - tan^2 A}$; $\tan A = \frac{2 \tan \frac{A}{2}}{1 - tan^2 \frac{A}{2}}$

Example: Given $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{9}{41}$, find the value of $\sin(\alpha - \beta)$, α, β being acute angles.

Solution.

$cos^2 \alpha = 1 - sin^2 \alpha = 1 - \frac{9}{25} = \frac{16}{25}$

$\cos\alpha = \frac{4}{5}$. Also, $sin^2\beta = 1 - cos^2\beta = 1 - \frac{81}{1681} = \frac{1600}{1681}$

$\sin\beta = \frac{40}{41}$. Therefore, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$= \frac{27}{205} - \frac{150}{205} = \frac{-133}{205}$

Example. Show that $\sin 105 + \cos 105 = \cos 45$

Solution.

$\sin 105 + \cos 105 = \sin(60 + 45) + \cos(60 + 45)$

$= \sin 60 \cos 45 + \cos 60 \sin 45 + (\cos 60 \cos 45 - \sin 60 \sin 45)$

$= \left(\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} = \cos 45$

Example. Solve for the value of $\tan 75$ and show that $\tan 75 + \cot 75 = 4$.

Solution.

$\tan 75 = \tan(45 + 30) = \frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30}$

$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}$

$\tan 75 + \cot 75 = 2 + \sqrt{3} + \frac{1}{2+\sqrt{3}} = 2 + \sqrt{3} + \frac{2-\sqrt{3}}{2+\sqrt{3}(2-\sqrt{3})} = 2 + \sqrt{3} + \frac{2-\sqrt{3}}{4-3} = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$

Assessment:

1. Solve for the value of $\cot 60^0 \tan 30^0 + sec^2 45^0$
2. Show that $\tan\theta + \cot\theta = \sec\theta\, cosec\theta$
3. Show that $\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta}{1-\sin\theta}$
4. Given that $u = \frac{1+\sin\theta}{\cos\theta}$ Show that $\tan\theta = \frac{u^2-1}{2u}$
5. Simplify $\sqrt{\frac{1-cos^2\theta}{sec^2\theta-1}}$
6. Show that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = cosec\,\theta + \cot\theta$
7. Given that $\sin\theta = \frac{-12}{13}$ solve the other trigonometric ratios of θ.
8. Show that $\frac{\sin A+\sin 3A}{\cos A+\cos 3A} = \tan 2A$
9. Show that $\frac{\cos 13+\sin 13}{\cos 13-\sin 13} = \tan 58$
10. Solve the equation $4sin^2x + \sin x - \frac{1}{2} = 0$ for $0^0 \le x \le 360^0$.

11. Solve the equation $6cos^2\theta + 10\sin\theta - 4 = 0$ for $0^0 \leq \theta \leq 360^0$.
12. Solve the equation $5cos^2\theta + 3\sin\theta - 3 = 0$ for $0^0 \leq \theta \leq 360^0$.

CHAPTER THIRTEEN

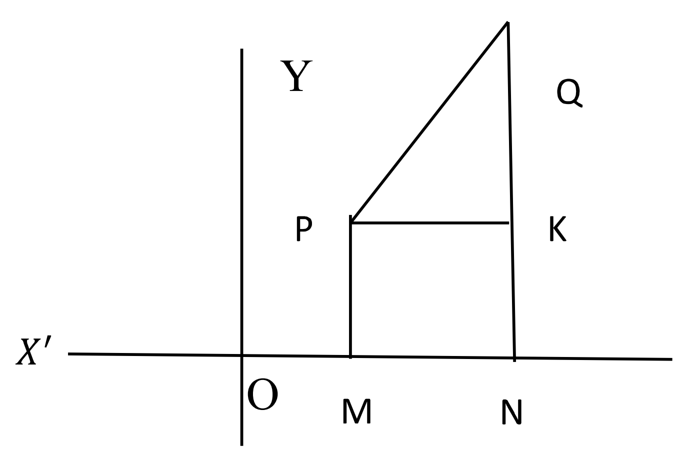
GEOMETRY

13.1 Introduction

Coordinate geometry or analytical geometry is that branch of mathematics that solves geometrical problems using algebra.

Coordinate system are used to represent points in a plane by means of a pair of real numbers called coordinates

13.2 Distance Between two Points



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points.

Draw perpendicular PM and QN on the x – axis where PK is perpendicular to QN

$x_1 = OM, y_1 = PM, x_2 = ON, y_2 = QN$. From the right angled triangle we have $PQ^2 = PK^2 + QK^2$

$PQ^2 = MN^2 + (QN - KN)^2 = (ON - OM)^2 + (QN - PM)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Distance of any point $P(x_1, y_1)$ from the origin is $\sqrt{{x_1}^2 + {y_1}^2}$

SECTION FORMULAS

To find the coordinates of the point which divides the joins of two given points (x_1, y_1) and (x_2, y_2) in the ratio $m_1 : m_2$

Internally, let P and Q be the two given points with coordinates (x_1, y_1) and (x_2, y_2) and R be the point that divides line PQ in the ratio $m_1 : m_2$ internally. Draw perpendiculars PL, RM, QN on the x – axis and take RT parallel to x – axis meeting QN in T and LP produced in K.

From similar triangles KPR and TQR, we have

$\frac{KR}{RT} = \frac{PR}{RQ} = \frac{m_1}{m_2}$ where

$KR = LM = OM - OL = x - x_1$; $RT = MN = ON - OM = x_2 - x$

$\frac{x-x_1}{x_2-x} = \frac{m_1}{m_2}$ This implies that $x = \frac{m_1x_2+m_2x_1}{m_1+m_2}$

Similarly, $\frac{PK}{TQ} = \frac{PR}{RQ} = \frac{m_1}{m_2}$ This implies that $y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$.

The coordinates of R are

$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$

When R is in the middle of PQ, We have $x = \frac{x_1+x_2}{2}$

and $y = \frac{y_1+y_2}{2}$

Externally,

Let R divide PQ externally, in the ratio $m_1 : m_2$.PL, QN, RN are perpendicular on the x – axis.

For KPR and TQR, we have $\frac{KR}{TR} = \frac{PR}{QR} = \frac{m_1}{m_2}$ where

$KR = LM = OM - OL = x - x_1$; $TR = NM = OM - ON = x - x_2$

$\frac{x-x_1}{x-x_2} = \frac{m_1}{m_2}$ This implies that $x = \frac{m_1x_2-m_2x_1}{m_1-m_2}$

Similarly, $\frac{PK}{TQ} = \frac{PR}{RQ} = \frac{m_1}{m_2}$ This implies that $y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$.

The coordinates of R are

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

Example 1. Solve for the distance between the points (4, -7) and (-1, 5)

Solution.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(4 - (-1))^2 + (5 - (-7))^2}$$

$$d = \sqrt{(5)^2 + (12)^2} = \sqrt{169} = 13$$

Example 2. Solve for the points that divides line of coordinates (2, 1) and (3, 5) externally in the ratio 2 : 3, the point lying towards the point (3, 5).

Solution: Let R(x, y) be the required point, then we have $m_1 = 2, m_2 = 3$

Using $x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} = \frac{2\times3-3\times2}{2-3} = 0$

$$y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2} = \frac{2\times5-3\times1}{2-3} = \frac{10-3}{-1} = -7$$

R has the (0, 7)

Example 3. Show that the points (6, 6), (2, 3), (4, 7) are the vertices of a right angled triangle

Solution. Let A (6, 6), (2, 3), (4, 7) are the vertices with three points

$$AB = \sqrt{(6-2)^2 + (6-3)^2} = \sqrt{16+9} = 5$$

$$BC = \sqrt{(2-4)^2 + (3-7)^2} = \sqrt{4+16} = \sqrt{20}$$

$$CA = \sqrt{(4-6)^2 + (7-6)^2} = \sqrt{4+1} = \sqrt{5}$$

$$25 = AB^2 = 20 + 5 = BC^2 + CA^2$$

Example 4. Show that the points (2, 6), (5, 1), (0, -2), (-3, 3) are the vertices of a square

Solution. Let the four given points be A (2, 6), B (5, 1), C (0, -2), D (-3, 3).

$$AB = \sqrt{(2-5)^2 + (6-1)^2} = \sqrt{9+25} = \sqrt{34}$$

$$BC = \sqrt{(5-0)^2 + (1+2)^2} = \sqrt{25+9} = \sqrt{34}$$

$$CD = \sqrt{(0+3)^2 + (-2-3)^2} = \sqrt{9+25} = \sqrt{34}$$

$$DA = \sqrt{(-3-2)^2 + (3-6)^2} = \sqrt{25+9} = \sqrt{34}$$

$$AB = BC = CD = DA$$

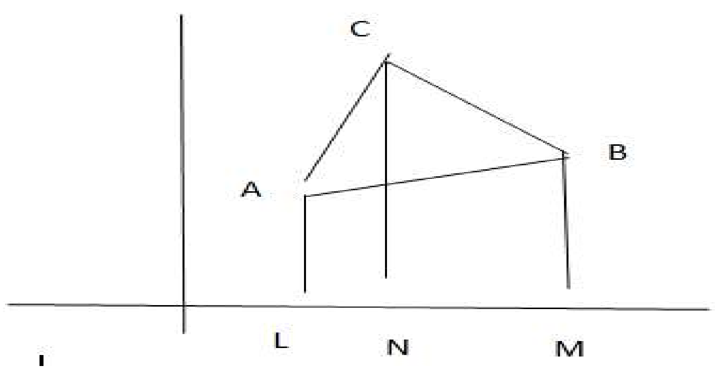
$$AC^2 = (2-0)^2 + (6+2)^2 = 4 + 64 = 68$$

$$AC^2 = 34 + 34 = AB^2 + BC^2$$

13.3 Area of a Triangle

To find the area of a triangle, coordinates of the vertices being given

ABC is the given triangle, whose vertices A, B, C have co-ordinates $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3). Draw perpendicular AL, BM, CN on the x – axis If Δ denotes the area of the triangle ABC, we have



$\Delta = trapezium\ ALNC + trapezium\ CNMB - trapezium\ ALMB$

Since the area of the trapezium is given by

$\frac{1}{2}(sum\ of\ parallel\ sides) \times (distance\ between\ them)$

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

When the points A, B, C are collinear then $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

Example. Solve for the area of the triangle whose vertices are $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$

Solution: The area of the triangle is given as $= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$$= \frac{1}{2}[at_1^2(2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_3^2(2at_1 - 2at_2)]$$

$$= a^2[t_1^2(t_2 - t_3) + t_2^2(t_3 - t_1) + t_3^2(t_1 - t_2)]$$

$$= -a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$$

Example. Show that the three points (1, 4), (3, -2) and (-3, 16) are collinear.

Solution. When the area of the triangle is equal to zero the three points are collinear

Area is expressed as